PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 100, Number 1, May 1987

ISOLATION THEOREM FOR PRODUCTS OF LINEAR FORMS

T. W. CUSICK

ABSTRACT. A theorem of Cassels and Swinnerton-Dyer about products of three linear forms with real coefficients is generalized to products of any number of linear forms.

1. Introduction. In this paper we generalize some results of Cassels and Swinnerton-Dyer [**3**]. Suppose $f(x_1, \ldots, x_n)$ is a product of $n \geq 3$ linear forms with real coefficients. Our first theorem is an isolation theorem for these forms $f(x_1, \ldots, x_n)$. For a discussion of the significance of isolation theorems in the geometry of numbers see Cassels [**2**, pp. 264–265]; for examples and applications of isolation theorems see [**2**, pp. 286–298].

In order to state our theorem we need to define an ε-neighborhood of the form f. For any $\varepsilon > 0$, an ε-neighborhood of f is the set of all products of n linear forms such that the coefficients of the linear forms are within ε of the corresponding coefficients of the linear forms in f. Any set which contains some ε-neighborhood of f will be called a neighborhood of f.

THEOREM 1. *Let $f(x_1, \ldots, x_n)$ be the product of $n \geq 3$ linear forms with real coefficients. Suppose that f has integer coefficients and that $f = 0$ only when all the x_i are 0. Let (δ_1, δ_2) be any open interval. Then there is a neighborhood of f such that all forms in the neighborhood which are not multiples of f itself take some value in the interval (δ_1, δ_2) for some integer values of the variables $x_1, \ldots, x_n$.*

It is well known (see [**2**, pp. 285–286]) that the conditions imposed on the product f of linear forms in Theorem 1 imply that f is equal to an integer times the product of all the n conjugates of one linear form whose coefficients are algebraic integers in some totally real algebraic number field of degree n. Thus Theorem 1 is really a statement about norm forms. Of course any norm form f satisfies $\inf|f| > 0$, where the infimum is taken over all integers $x_1, \ldots, x_n$ not all zero. It is a notorious unsolved problem [**2**, pp. 260–264] to decide whether norm forms are the only products of $n \geq 3$ linear forms with this property. For $n = 2$, this problem has a negative answer because we can find a binary quadratic form

$$f(x, y) = (x + \theta_1 y)(x + \theta_2 y) = x^2 + \beta xy + \gamma y^2$$

with γ irrational such that $\inf|f(x, y)| \geq 1$ for x, y not both zero. Such forms exist as long as $\beta^2 - 4\gamma \geq 9$, and indeed there are uncountably many with $\beta^2 - 4\gamma = 9$ (see Cassels [**1**, Lemma 14, pp. 38–39]). Also, the analog of Theorem 1 is false for $n = 2$, for instance if we consider the form $x^2 - 3y^2$. A certain weaker isolation theorem [**2**, pp. 287–289] is valid for $n = 2$.

Received by the editors September 26, 1985 and, in revised form, March 25, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 10E15; Secondary 10C10, 10F99.
Key words and phrases. Linear forms, norm forms.

The case $n = 3$ of Theorem 1 was proved by Cassels and Swinnerton-Dyer [**3**, Theorem 2, p. 74]. Skubenko [**4**, **5**] also claims a proof of Theorem 1, but his argument is difficult to follow (in particular, when he uses Lemma 3, Corollary in [**4**] to prove the theorem of that paper, he needs to show, in his notation, that $H^2\varepsilon^{1/2} \to 0$ as $\varepsilon \to 0$, but this is not proved). In any case, the proof of Theorem 1 given here is simpler than that of Skubenko; indeed, our proof is along the lines of the original proof of the case $n = 3$ [**3**, pp. 78–79], but with one additional idea.

2. Proof of Theorem 1. We first require a generalization of Lemma 1 of [**3**].

LEMMA 1. *Fix $m \geq 2$. Let $A = [a_{ij}]$ be an m by m matrix of real numbers with $\det A \neq 0$. Suppose not all of the numbers $a_{1j}a_{11}^{-1}$ $(j = 2, 3, \ldots, m)$ are rational. Then given any $\tau > 0$ there exists a σ, depending on τ and on the a_{ij}, such that for any λ there are integers $u_1, \ldots, u_m$ such that*

$$|u_1a_{11} + u_2a_{12} + \cdots + u_ma_{1m} - \lambda| < \tau$$

and

$$|u_1a_{i1} + u_2a_{i2} + \cdots + u_ma_{im}| \leq \sigma \quad \text{for } i = 2, 3, \ldots, m.$$

PROOF. This is an easy consequence of Kronecker's theorem on Diophantine approximation.

In order to apply Lemma 1 in the proof of Theorem 1, we shall need the following lemma about units in totally real fields. If α is a number in an algebraic number field of degree d, we let $\alpha = \alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(d)}$ denote the conjugates of α. The elegant proof of Lemma 2, due to Swinnerton-Dyer, replaces a clumsier one of the author.

LEMMA 2. *Suppose K is a totally real algebraic number field of degree $r \geq 3$ and $\chi_1, \ldots, \chi_{r-1}$ are any multiplicatively independent units in K. Then for any $i \neq j$, $1 \leq i, j \leq r$, such that $|\chi_1^{(i)}| \neq |\chi_1^{(j)}|$, the $r - 2$ numbers*

$$\frac{\log|\chi_k^{(i)}/\chi_k^{(j)}|}{\log|\chi_1^{(i)}/\chi_1^{(j)}|} \qquad (k = 2, 3, \ldots, r-1)$$

are not all rational.

PROOF. We suppose that for some fixed i, j and rational integers $p(k), q(k) \neq 0$ we have

$$\frac{\log|\chi_k^{(i)}/\chi_k^{(j)}|}{\log|\chi_1^{(i)}/\chi_1^{(j)}|} = \frac{p(k)}{q(k)} \qquad (k = 2, 3, \ldots, r-1); \tag{1}$$

we shall deduce a contradiction. We start with some simplifications, none of which alter the multiplicative independence of the χ_k. First we square each χ_k; then (1) still holds and we can remove the absolute value signs in (1) since every χ_k is totally positive. Now we define units

$$y_k = \chi_k^{q(k)}\chi_1^{-p(k)} \qquad (k = 2, 3, \ldots, r-1); \tag{2}$$

since $q(k) \neq 0$, the y_k $(k = 2, 3, \ldots, r-1)$ are multiplicatively independent. Now (1) gives

$$\log(y_k^{(i)}/y_k^{(j)}) = q(k)\log(\chi_k^{(i)}/\chi_k^{(j)}) - p(k)\log(\chi_1^{(i)}/\chi_1^{(j)}) = 0,$$

whence $y_k^{(i)} = y_k^{(j)}$ since y_k is totally positive.

Now let L be the subfield of K made up of those b such that $b^{(i)} = b^{(j)}$; since L is a proper subfield of K, it has degree at most $\frac{1}{2}r$. But (2) gives $r-2$ multiplicatively independent units of L, so we obtain

$$r - 2 \leq \tfrac{1}{2}r - 1$$

by Dirichlet's unit theorem. This contradiction proves Lemma 2.

As we saw in the Introduction, the form f in Theorem 1 is essentially the product of n linear forms whose coefficients lie in some totally real algebraic number field K of degree n. In order to simplify the notation, we take $n = 4$ for the rest of this section. Our arguments in the rest of the section clearly apply for all $n \geq 4$, and the case $n = 3$ is easier.

Thus we may suppose

$$f = L_1 L_2 L_3 L_4,$$

where the L_i are conjugate linear forms in x_1, x_2, x_3, x_4 with coefficients in the conjugate fields of K. If we define

$$L_i^* = \sum_{j=1}^{4} \varepsilon_{ij} L_j \qquad (1 \leq i \leq 4,\ \varepsilon_{jj} = 1 \text{ for } 1 \leq j \leq 4)$$

and

$$f^* = (1 + \varepsilon_0) L_1^* L_2^* L_3^* L_4^*,$$

then the forms f^* make up a neighborhood of f (in the sense defined in the Introduction) if we put bounds on ε_0 and the twelve ε_{ij} $(i \neq j)$. Note f^* is a multiple of f if and only if the ε_{ij}, $i \neq j$, are all zero.

We may suppose the conjugate fields of K are ordered so that if $L_1(x_1, x_2, x_3, x_4) = \xi$, then

$$L_j(x_1, x_2, x_3, x_4) = \xi^{(j)} \qquad (1 \leq j \leq 4).$$

The set of all values taken on by L_1 for integral x_i is a module, and we fix three positive independent units χ_i $(1 \leq i \leq 3)$ in the coefficient ring of this module such that $\chi_i^{(j)} > 0$ for $1 \leq i \leq 3$ and $1 \leq j \leq 4$. If $\mu = \chi_1^p \chi_2^q \chi_3^r$, this implies that for suitable integers y_i we have

$$L_j(u_1, y_2, y_3, y_4) = \mu^{(j)} \xi^{(j)} \qquad (1 \leq j \leq 4).$$

We now prove a generalization of Lemma 2 of [**3**].

LEMMA 3. *Given any $\omega > 0$ there exists a C, depending on ω and the units χ_i, with the following property: If Ψ is given, $0 < \Psi < 1$, then there exist integers p, q, r, depending on ω and Ψ, such that the unit $\theta = \chi_1^p \chi_2^q \chi_3^r$ satisfies*

$$\omega\theta < |\theta - \Psi\theta^{(2)}| < 2\omega\theta \tag{3}$$

and

$$\Psi^{1/2}\theta^{(i)} < C\theta^{(j)} \qquad (1 \leq i, j \leq 4,\ (i,j) \neq (2,1)). \tag{4}$$

PROOF. We apply Lemma 1 with $m = 3$ and

$$a_{1j} = \log(\chi_j/\chi_j^{(2)}), \quad a_{2j} = \log \chi_j \chi_j^{(2)}, \quad a_{3j} = \log \chi_j \chi_j^{(2)} \chi_j^{(3)} \qquad (1 \leq j \leq 3).$$

The matrix $A = [a_{ij}]$ is clearly a nonzero multiple of the regulator matrix of F, so $\det A \neq 0$. Also, by Lemma 2 with $r = 4$, the numbers $a_{12}a_{11}^{-1}$ and $a_{13}a_{11}^{-1}$ are not both rational, so the hypotheses of Lemma 1 are satisfied. (We can assume $a_{11} \neq 0$ by renumbering conjugates.) We choose

$$\lambda = \log \Psi, \qquad \tau = \log(1+\omega)$$

in Lemma 1 and we may suppose $\omega < 1$. We take $p = u_1, q = u_2, r = u_3$, where u_1, u_2, u_3 are the integers obtained in Lemma 1; then Lemma 1 gives

$$1 - \omega < (1+\omega)^{-1} < \Psi\theta^{(2)}\theta^{-1} < 1 + \omega \tag{5}$$

and

$$c^{-1} \leq \theta\theta^{(2)} \leq c, \qquad c^{-1} \leq \theta\theta^{(2)}\theta^{(3)} \leq c, \tag{6}$$

where $c = e^{\sigma}$ depends only on ω and the χ_i.

It follows immediately from (5) that

$$|\theta - \Psi\theta^{(2)}| < \omega\theta \tag{7}$$

and (5) and (6) together imply

$$(c^{-1}(1+\omega)^{-1}\Psi)^{1/2} < \theta < (c(1+\omega)\Psi)^{1/2}, \tag{8}$$

$$(c^{-1}(1+\omega)^{-1}\Psi^{-1})^{1/2} < \theta^{(2)} < (c(1+\omega)\Psi^{-1})^{1/2}, \tag{9}$$

$$c^{-2} \leq \theta^{(3)} \leq c^2, \qquad c^{-1} \leq \theta^{(4)} \leq c, \tag{10}$$

since $\theta\theta^{(2)}\theta^{(3)}\theta^{(4)} = 1$. Since c is independent of Ψ, (8), (9), and (10) imply that (4) holds for a constant C depending only on ω and the χ_i.

We may suppose $\omega < 2/3$, and then we may replace Ψ by $2\Psi/(2-3\omega)$ and ω by $\omega/(2-3\omega)$ in the lemma. Now (7) implies (3) and we can change the value of C so that (4) still holds. This completes the proof of Lemma 3.

We can now prove Theorem 1. It is enough to show that given any $\delta > 0$, the inequality

$$0 < |f^*| = |L_1^* L_2^* L_3^* L_4^*| < \delta \tag{11}$$

is solvable for any form f^*, not a multiple of f, in some neighborhood of f: for if f^* takes on some value δ_0, then it also takes on all values $m^4\delta_0$ for integer m, and we need only choose δ_0 so small that

$$\delta_1 < m^4\delta_0 < (m+1)^4\delta_0 < \delta_2$$

for some integer m.

Plainly we may suppose $\varepsilon_0 = 0$ without loss of generality, and we shall also assume

$$\varepsilon_{12} = \max_{i \neq j} |\varepsilon_{ij}| > 0;$$

this is one of 24 possible cases for the maximum of ε_{ij} $(i \neq j)$, sign being taken into account, and all of these cases can be handled in the same way.

In order to satisfy (11) we shall take

$$L_j = (\chi_1^p \chi_2^q \chi_3^r)^{(j)} \xi^{(j)} \qquad (1 \leq j \leq 4), \tag{12}$$

where ξ is a fixed value of L_1 and p, q, r are chosen so that L_1^* is small compared to L_1, but L_j^* is similar in size to L_j for $j = 2, 3, 4$. We fix ξ so that $\xi\xi^{(2)} < 0$ and define

$$\Psi = -\varepsilon_{12}\xi^{-1}\xi^{(2)} > 0. \tag{13}$$

Now we have $\Psi \to 0$ as $\varepsilon_{12} \to 0$.

We choose a small ω and take ε_{12} so small that $\Psi < 1$, so Lemma 3 applies and we define the L_j by (12), where p, q, r are the integers determined in Lemma 3. Now (3) gives

$$\omega\theta|\xi| < |\theta\xi + \varepsilon_{12}\theta^{(2)}\xi^{(2)}| < 2\omega\theta|\xi|,$$

so (4) and (13) imply

$$\begin{aligned}|L_1^*| &\geq |\theta\xi + \varepsilon_{12}\theta^{(2)}\xi^{(2)}| - |\varepsilon_{13}\theta^{(3)}\xi^{(3)}| - |\varepsilon_{14}\theta^{(4)}\xi^{(4)}| \\ &> \omega\theta|\xi| - C|\xi\xi^{(3)}/\xi^{(2)}|\Psi^{1/2}\theta - C|\xi\xi^{(4)}/\xi^{(2)}|\Psi^{1/2}\theta > 0\end{aligned}$$

provided ε_{12} (and so Ψ) is small enough. Also (3), (4), and (13) give

$$\begin{aligned}|L_1^*| &< |\theta\xi + \varepsilon_{12}\theta^{(2)}\xi^{(2)}| + |\varepsilon_{13}\theta^{(3)}\xi^{(3)}| + |\varepsilon_{14}\theta^{(4)}\xi^{(4)}| \\ &< 2\omega\theta|\xi| + \varepsilon_{12}^{1/2}|\xi^{-1}\xi^{(2)}|^{-1/2} < 3\omega\theta|\xi|\end{aligned}$$

if ε_{12} is small enough. By a similar but simpler argument

$$0 < |L_j^*| < 2\theta^{(j)}|\xi^{(j)}| \qquad (j = 2, 3, 4)$$

if ε_{12} is small enough; so we finally obtain

$$0 < |L_1^* L_2^* L_3^* L_4^*| < 24\omega|\xi\xi^{(2)}\xi^{(3)}\xi^{(4)}|.$$

Since we may choose ω as small as we wish, the proves (11) and so Theorem 1.

REFERENCES

1. J. W. S. Cassels, *An introduction to diophantine approximation*, Cambridge Univ. Press, 1957.
2. ——, *An introduction to the geometry of numbers*, Springer-Verlag, Berlin, 1959.
3. J. W. S. Cassels and H. P. F. Swinnerton-Dyer, *On the product of three homogeneous linear forms and indefinite ternary quadratic forms*, Philos. Trans. Roy. Soc. London Ser A **248** (1955), 73–96.
4. B. F. Skubenko, *On the product of n linear forms in n variables*, Trudy Mat. Inst. Steklov. **158** (1981), 175–179. (Russian)
5. ——, *Isolation theorems for decomposable forms over totally real algebraic number fields of degree* $n \geq 3$, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Int. Steklov. (LOMI) **112** (1981), 167–171. (Russian)

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK, BUFFALO, NEW YORK 14214