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AN INEQUALITY FOR EIGENVALUES OF STURM-LIOUVILLE PROBLEMS

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ABSTRACT. By means of simple transformations, inequalities for eigenvalues corresponding to different boundary conditions are derived.

1. Let $\rho(x) > 0$ be a continuous function and consider the following eigenvalue problems:

$$\phi''(x) + \lambda\rho(x)\phi(x) = 0 \quad \text{in } (-1,1), \qquad \phi(-1) = \phi(1) = 0, \tag{1.1}$$

and

$$\psi''(x) + \mu\rho(x)\psi(x) = 0 \quad \text{in } (-1,1), \qquad \psi'(-1) = \psi'(1) = 0. \tag{1.2}$$

It is well known that there exist two countable sequences of eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$ and $0 = \mu_1 < \mu_2 < \cdots$ both tending to infinity as $n \to \infty$. According to Poincaré's principle they may be characterized as

$$\lambda_n = \inf_{L'_n} \max_{v \in L'_n} R[v] \quad \text{and} \quad \mu_n = \inf_{L_n} \max_{v \in L_n} R[v], \tag{1.3}$$

where

$$R[v] := \int_{-1}^{1} v'^2\,dx \Big/ \int_{-1}^{1} v^2 \rho\,dx$$

is the Rayleigh quotient, $L'_n \subset H_0^1(-1,1)$ and $L_n \subset H^1(-1,1)$ are n-dimensional function spaces. In (1.3), equality holds if and only if v is the nth eigenfunction.

2. From (1.3) it follows immediately that

$$\mu_n < \lambda_n. \tag{2.1}$$

The aim of this note is to sharpen inequality (2.1) for a special class of mass densities ρ.

3. Let us start with two elementary lemmas. The first observation is essentially due to [**3**].

LEMMA 1. *Let ϕ_1 and ϕ_n be the first and the nth eigenfunctions of* (1.1). *Then $v_n(x) = \phi_n(x)/\phi_1(x)$ is the nth eigenfunction of*

$$\{\phi_1^2 v'_n\}' + \tilde{\omega}\phi_1^2\rho v_n = 0 \quad \text{in } (-1,1), \qquad v'_n(-1) = v'_n(1) = 0, \tag{3.1}$$

with the corresponding eigenvalue $\tilde{\omega}_n = \lambda_n - \lambda_1$.

PROOF. Since ϕ_k has the expansion $\phi_k(x) = \phi'_k(-1)(x+1) + o((x+1)^2)$, $\phi'_k(-1) \neq 0$, it follows that

$$v'(x) = \frac{\phi'_n(x)\phi_1(x) - \phi_n(x)\phi'_1(x)}{\phi_1^2(x)} = O(x+1),$$

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which implies $v'(-1) = 0$. Similarly we have $v'(1) = 0$. The remaining part is verified by a straightforward computation.

The next result is a slightly modified version of a lemma by Payne and Weinberger [**2**].

LEMMA 2. *Let v_n be the nth eigenfunction of* (3.1). *Then $w_{n-1} := v_n'\phi_1$ is the $(n-1)$st eigenfunction of*

$$\left(\frac{w'}{\rho}\right)' - \frac{1}{\rho}\left[\frac{2\phi_1'^2}{\phi_1^2} + \frac{\phi_1'\rho'}{\phi_1\rho}\right] w + \nu w = 0 \quad \text{in } (-1,1), \qquad w(-1) = w(1) = 0, \tag{3.2}$$

with the corresponding eigenvalue $\nu_{n-1} = \lambda_n - 2\lambda_1$.

The proof of (3.2) follows from a straightforward computation. In view of Poincaré's variational principle we have

$$\nu_{n-1} = \inf_{L'_{n-1}} \max_{v\in L'_{n-1}} \left\{\int_{-1}^{1} \frac{v'^2}{\rho}\,dx + \int_{-1}^{1}\left[\frac{2\phi_1'^2}{\phi_1^2} + \frac{\phi_1'\rho'}{\phi_1\rho}\right]\frac{v^2}{\rho}\,dx\right\} \Big/ \int_{-1}^{1} v^2\,dx, \tag{3.3}$$

where L'_{n-1} is an $(n-1)$-dimensional space of functions in $H_0^1(-1,1)$ for which (3.3) makes sense. If the assumption

$$\frac{2\phi_1'^2}{\phi_1^2} + \frac{\phi_1'\rho'}{\phi_1\rho} \geq 0 \tag{A}$$

holds, then we have by (3.3)

$$\nu_{n-1} \geq \omega_{n-1}, \tag{3.4}$$

where ω_{n-1} is the $(n-1)$st eigenvalue of

$$\left(\frac{u'}{\rho}\right)' + \omega u = 0 \quad \text{in } (-1,1), \qquad u(-1) = u(1) = 0. \tag{3.5}$$

LEMMA 3. *$\omega_{n-1} = \mu_n$, where μ_n is the nth eigenvalue of* (1.2).

PROOF. Put $u'/\rho = \psi$. Then $\psi' = -\omega u$ and $\psi'' + \omega\rho\psi = 0$, which proves the assertion.

We are now ready to establish our main result.

THEOREM 1. *Under the assumptions*
(i) $\rho(-x) = \rho(x)$, and
(i) $\rho(x)$ increasing in $(-1,0)$,
we have $\lambda_n - 2\lambda_1 \geq \mu_n$.

PROOF. We first consider the case where ρ is differentiable. In view of assumption (i), ϕ_1 is symmetric with respect to $x = 0$. Moreover $|\phi_1(x)|$ is concave, hence $\phi_1'\rho'/\phi_1 \geq 0$ in $(-1,1)$. Thus assumption (A) is satisfied. The assertion now follows from (3.4) together with Lemmas 2 and 3. If ρ is not differentiable we approximate ρ by means of differentiable functions $\{\rho_n\}_{n=1}^{\infty}$ and use the fact that λ_n and μ_n depend continuously on ρ [**1**, p. 418].

4. Consider the eigenvalue prolem with mixed boundary conditions

$$\phi''(x) + \Lambda\rho(x)\phi(x) = 0 \quad \text{in } (0,1), \qquad \phi(0) = \phi'(1) = 0. \tag{4.1}$$

It is easily seen that for the first eigenfunction ϕ_1 we have $\phi_1\phi_1' > 0$ in $(0,1)$. If

(iii) $\rho(x)$ is increasing in $(0,1)$,

then condition (A) is satisfied. The same arguments as for Theorem 1 yield

THEOREM 2. *Assume* (iii). *Then* $\Lambda_n - 2\Lambda_1 \geq \mu_n$.

We finally consider the more general eigenvalue problem

$$(\sigma(x)\phi'(x))' + \lambda\rho(x)\phi(x) = 0 \quad \text{in } (-1,1) \tag{4.2}$$

with either boundary conditions

$$\phi(-1) = \phi(1) = 0, \tag{4.3}$$

$$\phi(-1) = \phi'(1) = 0, \tag{4.4}$$

or

$$\phi'(-1) = \phi'(1) = 0. \tag{4.5}$$

If we introduce the new variable $t = \int_{-1}^{x}(ds/\sigma(s))$, (4.2) becomes

$$\ddot{\phi} + \lambda\sigma(x(t))\rho(x(t))\phi = 0. \tag{4.6}$$

A direct application of Theorems 1 and 2 yields obviously

THEOREM 3. *Let* $\lambda_n, \mu_n, \Lambda_n$ *be the eigenvalues of* (4.2) *with the boundary equations* (4.3), (4.4), *or* (4.5) *respectively. Under the assumptions*

(i)′ $\sigma\rho(-x) = \sigma\rho(x)$,

(ii)′ $\sigma\rho$ *increasing in* $(-1,0)$,

we then have $\lambda_n - 2\lambda_1 \geq \mu_n$.

Under the assumption

(iii)′ $\sigma\rho$ *increasing in* $(-1,1)$,

we then have $\Lambda_n - 2\Lambda_1 \geq \mu_n$.

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