# AN ELEMENTARY PROOF
# OF THE GROTHENDIECK INEQUALITY

## RON C. BLEI

ABSTRACT. An elementary proof of the Grothendieck inequality is given.

Since its appearance in [2, pp. 59–64] and reformulation in [3, pp. 277–280], Grothendieck's fundamental inequality has enjoyed several restatements and proofs within various frameworks of analysis (detailed accounts of which appear in [4]). The purpose of this note is to give an elementary and self-contained proof of the inequality: the argument below, an adaptation of the proof given in [1], requires knowing only that the expectation of a product of independent random variables equals the product of their expectations.

Let $\mathbf{R}^{\mathbf{N}}$ denote the space of sequences of real numbers with finitely many nonzero terms. $\mathbf{R}^{\mathbf{N}}$ will be equipped with the usual inner product,

$$\langle x, y \rangle = \sum_n x(n)y(n), \qquad x, y \in \mathbf{R}^{\mathbf{N}},$$

and Euclidean norm,

$$\|x\| = \langle x, x \rangle^{1/2}, \qquad x \in \mathbf{R}^{\mathbf{N}}.$$

$B$ will denote the unit ball in $\mathbf{R}^{\mathbf{N}}$, i.e. $B = \{x \in \mathbf{R}^{\mathbf{N}} : \|x\| \le 1\}$.

THEOREM (GROTHENDIECK'S INEQUALITY). *Let $(a_{mn})_{m,n=1}^{\infty}$ be an array of complex numbers which satisfies*

$$(1) \qquad \left| \sum_{m,n=1}^{N} a_{mn} s_m t_n \right| \le \max_{1 \le m,n \le N} |s_m| \, |t_n|$$

*for all sequences of complex numbers $(s_m)_{m=1}^{\infty}$, $(t_n)_{n=1}^{\infty}$, and all integers $N \ge 1$. Then, for all sequences of vectors in $\mathbf{R}^{\mathbf{N}}$, $(x_m)_{m=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$,*

$$(2) \qquad \left| \sum_{m,n=1}^{N} a_{mn} \langle x_m, y_n \rangle \right| \le K \max_{1 \le m,n \le N} \|x_m\| \, \|y_n\|$$

*for all $N \ge 1$ and some universal constant $K$.*

To start, define a real-valued function on $\mathbf{R}^{\mathbf{N}} \times \mathbf{R}^{\mathbf{N}}$ by

$$(3) \qquad A(x, y) = \prod_n (1 + x(n)y(n)), \qquad x, y \in \mathbf{R}^{\mathbf{N}},$$

and estimate

$$(4) \qquad |A(x, y)| \le e^{\sum \ln(1 + |x(n)y(n)|)} \le e^{\sum |x(n)y(n)|} \le e^{\|x\| \cdot \|y\|}.$$

LEMMA 1. *Suppose* $(a_{mn})_{m,n=1}^{\infty}$ *satisfies the hypothesis of the theorem above. Then, for all sequences of vectors* $(x_m)_{m=1}^{\infty}, (y_n)_{n=1}^{\infty}$ *in the unit ball of* $\mathbf{R}^{\mathbf{N}}$,

$$\left| \sum_{m,n=1}^{N} a_{mn} A(x_m, y_n) \right| \leq e$$

*for all* $N \geq 1$.

PROOF. Let $(Z_n)_{n=1}^{\infty}$ be a sequence of independent real-valued random variables on some probability space so that

$$(5) \qquad \mathbf{E}(Z_n) = 0, \quad \mathbf{E}(Z_n^2) = 1, \quad \text{and} \quad |Z_n| = 1 \quad \text{a.s. for all } n.$$

($\mathbf{E}$ denotes expectation; $(Z_n)_{n=1}^{\infty}$ could be taken as the usual system of Rademacher functions.) Given $x \in \mathbf{R}^{\mathbf{N}}$, define a random variable

$$F(x) = \prod_n (1 + ix(n)Z_n) \qquad (i = \sqrt{-1}),$$

and estimate (by (4))

$$(6) \qquad |F(x)| \leq \left( \prod_n (1 + x(n)^2) \right)^{1/2} \leq e^{\|x\|^2/2} \quad \text{almost surely.}$$

For any $x, y \in \mathbf{R}^{\mathbf{N}}$,

$$\mathbf{E}(F(x)\overline{F(y)}) = \prod_n \mathbf{E}(1 + ix(n)Z_n)(1 - iy(n)Z_n) \quad \text{(by independence)}$$

$$= \prod_n (1 + x(n)y(n)) \quad \text{(by (5))}$$

$$= A(x, y).$$

Therefore, for any $(x_m)_{m=1}^{\infty}, (y_n)_{n=1}^{\infty} \subset B$ and all $N \geq 1$,

$$\left| \sum_{m,n=1}^{N} a_{mn} A(x_m, y_n) \right| = \left| \sum_{m,n=1}^{N} a_{mn} \mathbf{E}(F(x_m)\overline{F(y_n)}) \right|$$

$$\leq \mathbf{E} \left| \sum_{m,n=1}^{N} a_{mn} F(x_m)\overline{F(y_n)} \right|$$

$$\leq e \quad \text{(by (1) and (6)).} \quad \text{Q.E.D.}$$

Next, expand the product on the right-hand side of (3):

$$(7) \quad A(x, y) = 1 + \langle x, y \rangle + \cdots + \sum_{n_1 > \cdots > n_J} x(n_1) \cdots x(n_J) y(n_1) \cdots y(n_J) + \cdots.$$

Let $\{E_J\}_{J=2}^{\infty}$ be an infinite partition of the natural numbers $\mathbf{N}$, so that each $E_J \subset \mathbf{N}$ is infinite. Let $W_J$ be the $J$-dimensional wedge in $\mathbf{N}^J$ given by

$$W_J = \{(n_1, \ldots, n_J) \in \mathbf{N}^J : n_1 > \cdots > n_J\},$$

and set up a one-to-one correspondence between $E_J$ and $W_J$, $J \geq 2$:

$$n \in E_J \leftrightarrow (n_1, \ldots, n_J) \in W_J.$$

Given an arbitrary $x \in B$, define a vector $\phi(x) = (\phi(x)(n))_{n \in \mathbf{N}}$ in $\mathbf{R}^{\mathbf{N}}$ by

$$\phi(x)(n) = x(n_1) \cdots x(n_J), \qquad n \in E_J, \ J = 2, \ldots,$$

and estimate

$$(8) \qquad \|\phi(x)\| = \left( \sum_{J=2}^{\infty} \sum_{n \in E_J} (x(n_1) \cdots x(n_J))^2 \right)^{1/2}$$

$$\leq \left( \sum_{J=2}^{\infty} \frac{1}{J!} \left( \sum_{n \in \mathbf{N}} x(n)^2 \right)^J \right)^{1/2} \leq (e-2)^{1/2} \equiv \delta < 1.$$

Write

$$\phi_\delta(x) = \phi(x)/\delta, \qquad x \in B,$$

and, by the estimate above, note that $\phi_\delta$ is a map from $B$ into $B$. Solving for $\langle x, y \rangle$ in (7), $x, y \in B$, we obtain

$$(9) \qquad \langle x, y \rangle = A(x,y) - 1 - \delta^2 \langle \phi_\delta(x), \phi_\delta(y) \rangle.$$

Therefore, applying (9) recursively, we obtain for each $J > 0$

$$(10) \qquad \langle x, y \rangle = \sum_{j=0}^{J} (-\delta^2)^j [A(\phi_\delta^j(x), \phi_\delta^j(y)) - 1] + (-\delta^2)^{J+1} \langle \phi_\delta^{J+1}(x), \phi_\delta^{J+1}(y) \rangle$$

($\phi_\delta^j$ denotes the $j$th iterate of $\phi_\delta$). Finally, letting $J \to \infty$ in (10), we deduce

LEMMA 2. *For all $x, y \in B$*

$$\langle x, y \rangle = \sum_{j=0}^{\infty} (-\delta^2)^j [A(\phi_\delta^j(x), \phi_\delta^j(y)) - 1].$$

PROOF OF GROTHENDIECK'S INEQUALITY. It suffices to establish (2) for $(x_m)_{m=1}^{\infty}, (y_n)_{n=1}^{\infty} \subset B$. By Lemmas 1 and 2, we estimate

$$\left| \sum_{m,n=1}^{N} a_{mn} \langle x_m, y_n \rangle \right| \leq \sum_{j=0}^{\infty} \delta^{2j} \left| \sum_{m,n=1}^{N} a_{mn} [A(\phi_\delta^j(x_m), \phi_\delta^j(y_n)) - 1] \right|$$

$$\leq \sum_{j=0}^{\infty} \delta^{2j} (e+1) = (e+1)/(3-e). \quad \text{Q.E.D.}$$

REFERENCES

1. R. C. Blei, *A uniformity property for $\Lambda(2)$ sets and Grothendieck's inequality*, Sympos. Math. **22** (1977), 321–336.
2. A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologique*, Bol. Soc. Mat. São Paulo **8** (1956), 1–79.
3. J. Lindenstrauss and A. Pelczynski, *Absolutely summing operators in $\mathcal{L}^p$-spaces and their applications*, Studia Math. **29** (1968), 275–326.
4. G. Pisier, *Factorization of linear operators and geometry of Banach spaces*, CBMS Regional Conf. Ser. in Math. vol. 60, Amer. Math. Soc., Providence, R.I., 1986.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF CONNECTICUT, STORRS, CONNECTICUT 06268