

AN ELEMENTARY PROOF OF THE GROTHENDIECK INEQUALITY

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ABSTRACT. An elementary proof of the Grothendieck inequality is given.

Since its appearance in [2, pp. 59–64] and reformulation in [3, pp. 277–280], Grothendieck's fundamental inequality has enjoyed several restatements and proofs within various frameworks of analysis (detailed accounts of which appear in [4]). The purpose of this note is to give an elementary and self-contained proof of the inequality: the argument below, an adaptation of the proof given in [1], requires knowing only that the expectation of a product of independent random variables equals the product of their expectations.

Let \mathbf{R}^N denote the space of sequences of real numbers with finitely many nonzero terms. \mathbf{R}^N will be equipped with the usual inner product,

$$\langle x, y \rangle = \sum_n x(n)y(n), \quad x, y \in \mathbf{R}^N,$$

and Euclidean norm,

$$\|x\| = \langle x, x \rangle^{1/2}, \quad x \in \mathbf{R}^N.$$

B will denote the unit ball in \mathbf{R}^N , i.e. $B = \{x \in \mathbf{R}^N : \|x\| \leq 1\}$.

THEOREM (GROTHENDIECK'S INEQUALITY). *Let $(a_{mn})_{m,n=1}^\infty$ be an array of complex numbers which satisfies*

$$(1) \quad \left| \sum_{m,n=1}^N a_{mn} s_m t_n \right| \leq \max_{1 \leq m,n \leq N} |s_m| |t_n|$$

for all sequences of complex numbers $(s_m)_{m=1}^\infty$, $(t_n)_{n=1}^\infty$, and all integers $N \geq 1$. Then, for all sequences of vectors in \mathbf{R}^N , $(x_m)_{m=1}^\infty$, $(y_n)_{n=1}^\infty$,

$$(2) \quad \left| \sum_{m,n=1}^N a_{mn} \langle x_m, y_n \rangle \right| \leq K \max_{1 \leq m,n \leq N} \|x_m\| \|y_n\|$$

for all $N \geq 1$ and some universal constant K .

To start, define a real-valued function on $\mathbf{R}^N \times \mathbf{R}^N$ by

$$(3) \quad A(x, y) = \prod_n (1 + x(n)y(n)), \quad x, y \in \mathbf{R}^N,$$

and estimate

$$(4) \quad |A(x, y)| \leq e^{\sum \ln(1 + |x(n)y(n)|)} \leq e^{\sum |x(n)y(n)|} \leq e^{\|x\| \cdot \|y\|}.$$

Received by the editors January 2, 1986 and, in revised form, March 20, 1986.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 26D15, 46C99; Secondary 47A68.

The author was partially supported by NSF Grant MCS-8301659.

LEMMA 1. Suppose $(a_{mn})_{m,n=1}^{\infty}$ satisfies the hypothesis of the theorem above. Then, for all sequences of vectors $(x_m)_{m=1}^{\infty}, (y_n)_{n=1}^{\infty}$ in the unit ball of \mathbf{R}^N ,

$$\left| \sum_{m,n=1}^N a_{mn} A(x_m, y_n) \right| \leq e$$

for all $N \geq 1$.

PROOF. Let $(Z_n)_{n=1}^{\infty}$ be a sequence of independent real-valued random variables on some probability space so that

$$(5) \quad \mathbf{E}(Z_n) = 0, \quad \mathbf{E}(Z_n^2) = 1, \quad \text{and} \quad |Z_n| = 1 \quad \text{a.s. for all } n.$$

(\mathbf{E} denotes expectation; $(Z_n)_{n=1}^{\infty}$ could be taken as the usual system of Rademacher functions.) Given $x \in \mathbf{R}^N$, define a random variable

$$F(x) = \prod_n (1 + ix(n)Z_n) \quad (i = \sqrt{-1}),$$

and estimate (by (4))

$$(6) \quad |F(x)| \leq \left(\prod_n (1 + x(n)^2) \right)^{1/2} \leq e^{\|x\|^2/2} \quad \text{almost surely.}$$

For any $x, y \in \mathbf{R}^N$,

$$\begin{aligned} \mathbf{E}(F(x)\overline{F(y)}) &= \prod_n \mathbf{E}(1 + ix(n)Z_n)(1 - iy(n)Z_n) \quad (\text{by independence}) \\ &= \prod_n (1 + x(n)y(n)) \quad (\text{by (5)}) \\ &= A(x, y). \end{aligned}$$

Therefore, for any $(x_m)_{m=1}^{\infty}, (y_n)_{n=1}^{\infty} \subset B$ and all $N \geq 1$,

$$\begin{aligned} \left| \sum_{m,n=1}^N a_{mn} A(x_m, y_n) \right| &= \left| \sum_{m,n=1}^N a_{mn} \mathbf{E}(F(x_m)\overline{F(y_n)}) \right| \\ &\leq \mathbf{E} \left| \sum_{m,n=1}^N a_{mn} F(x_m)\overline{F(y_n)} \right| \\ &\leq e \quad (\text{by (1) and (6)}). \quad \text{Q.E.D.} \end{aligned}$$

Next, expand the product on the right-hand side of (3):

$$(7) \quad A(x, y) = 1 + \langle x, y \rangle + \dots + \sum_{n_1 > \dots > n_J} x(n_1) \dots x(n_J) y(n_1) \dots y(n_J) + \dots$$

Let $\{E_J\}_{J=2}^{\infty}$ be an infinite partition of the natural numbers \mathbf{N} , so that each $E_J \subset \mathbf{N}$ is infinite. Let W_J be the J -dimensional wedge in \mathbf{N}^J given by

$$W_J = \{(n_1, \dots, n_J) \in \mathbf{N}^J : n_1 > \dots > n_J\},$$

and set up a one-to-one correspondence between E_J and W_J , $J \geq 2$:

$$n \in E_J \leftrightarrow (n_1, \dots, n_J) \in W_J.$$

Given an arbitrary $x \in B$, define a vector $\phi(x) = (\phi(x)(n))_{n \in \mathbf{N}}$ in $\mathbf{R}^{\mathbf{N}}$ by

$$\phi(x)(n) = x(n_1) \cdots x(n_J), \quad n \in E_J, \quad J = 2, \dots,$$

and estimate

$$(8) \quad \|\phi(x)\| = \left(\sum_{J=2}^{\infty} \sum_{n \in E_J} (x(n_1) \cdots x(n_J))^2 \right)^{1/2} \\ \leq \left(\sum_{J=2}^{\infty} \frac{1}{J!} \left(\sum_{n \in \mathbf{N}} x(n)^2 \right)^J \right)^{1/2} \leq (e-2)^{1/2} \equiv \delta < 1.$$

Write

$$\phi_\delta(x) = \phi(x)/\delta, \quad x \in B,$$

and, by the estimate above, note that ϕ_δ is a map from B into B . Solving for $\langle x, y \rangle$ in (7), $x, y \in B$, we obtain

$$(9) \quad \langle x, y \rangle = A(x, y) - 1 - \delta^2 \langle \phi_\delta(x), \phi_\delta(y) \rangle.$$

Therefore, applying (9) recursively, we obtain for each $J > 0$

$$(10) \quad \langle x, y \rangle = \sum_{j=0}^J (-\delta^2)^j [A(\phi_\delta^j(x), \phi_\delta^j(y)) - 1] + (-\delta^2)^{J+1} \langle \phi_\delta^{J+1}(x), \phi_\delta^{J+1}(y) \rangle$$

(ϕ_δ^j denotes the j th iterate of ϕ_δ). Finally, letting $J \rightarrow \infty$ in (10), we deduce

LEMMA 2. For all $x, y \in B$

$$\langle x, y \rangle = \sum_{j=0}^{\infty} (-\delta^2)^j [A(\phi_\delta^j(x), \phi_\delta^j(y)) - 1].$$

PROOF OF GROTHENDIECK'S INEQUALITY. It suffices to establish (2) for $(x_m)_{m=1}^{\infty}, (y_n)_{n=1}^{\infty} \subset B$. By Lemmas 1 and 2, we estimate

$$\left| \sum_{m,n=1}^N a_{mn} \langle x_m, y_n \rangle \right| \leq \sum_{j=0}^{\infty} \delta^{2j} \left| \sum_{m,n=1}^N a_{mn} [A(\phi_\delta^j(x_m), \phi_\delta^j(y_n)) - 1] \right| \\ \leq \sum_{j=0}^{\infty} \delta^{2j} (e+1) = (e+1)/(3-e). \quad \text{Q.E.D.}$$

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