CONVOLUTION EQUATIONS IN SPACES OF DISTRIBUTIONS SUPPORTED BY CONES

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ABSTRACT. We describe some examples of surjective convolutors on $\mathcal{D}'(\Gamma)$, for $\Gamma$ a closed convex cone in $\mathbb{R}^n$. We also give necessary and sufficient conditions on $S_1, \ldots, S_m$ in $\mathcal{S}'(T)$ to be generators of the whole convolution algebra $\mathcal{S}'(\Gamma)$.

0. Introduction. Let $\Gamma \subset \mathbb{R}^n$ be a closed convex cone, whose vertex is the origin, and which does not contain any straight line. Denote by $\mathcal{D}'(\Gamma)$ the space of distributions supported in $\Gamma$; it is well known that $\mathcal{D}'(\Gamma)$ is a convolution algebra, so if $S \in \mathcal{E}'(\mathbb{R}^n)$ is a distribution with compact support contained in $\Gamma$, then $S * \mathcal{D}'(\Gamma) \subseteq \mathcal{D}'(\Gamma)$ (see, e.g., [6, p. 33]). By now, a complete characterization (in terms of the decrease of $\hat{S}$, the Fourier transform of $S$) of those $S$ for which $S * \mathcal{D}'(\Gamma) = \mathcal{D}'(\Gamma)$ is well known (see, e.g., [4, pages 356 and following]). On the other hand, in a recent paper, [8] a characterization of a completely different nature is given, at least for $n = 1$ (and hence $\Gamma = [0, +\infty)$); this characterization, unfortunately, is of difficult application, as it requires the test of the singular support of $S * u$ for an extremely large class of distributions $u$ and, as a consequence, no nontrivial examples are given in [8].

The purpose of this paper is twofold: first, in §1, we show that the condition of Shambayati and Zielezny [8] can be replaced by an equivalent one, in which only compactly supported distributions are used, and which has the advantage of being quite easy to check; in particular we are able to use our condition to provide several examples of distributions $S$ in $\mathcal{E}'(\mathbb{R})$ for which $S * \mathcal{D}'([0, +\infty)) = \mathcal{D}'([0, +\infty))$. One can now use this characterization, together with the well-known results of Hörmander on hyperbolic convolution equations, to deduce nontrivial properties of the Fourier transform of those invertible distributions $S \in \mathcal{E}'(\mathbb{R})$ for which $\text{sing supp}(S) = \{0\}$, and of other classes of examples. In this section we also prove that the original characterization provided in [8] can indeed be extended to the case $n > 1$; here, however, as we show with a simple example taken from the theory of constant coefficients partial differential operators, the condition cannot be eased.

In §2, finally, we deal with the following related problem: let $\mathcal{S}'(\Gamma)$ denote the space of tempered distributions, with support contained in the cone $\Gamma \subset \mathbb{R}^n$, and let $S_1, \ldots, S_m$ be $m$ distributions in $\mathcal{S}'(\Gamma)$; we provide a necessary and sufficient condition on $(S_1, \ldots, S_m)$ in order to ensure the existence of $m$ distributions $E_1, \ldots, E_m \in \mathcal{S}'(\Gamma)$ such that $S_1 * E_1 + \cdots + S_m * E_m = \delta$; for $m = 1$ the condition was well known (see [7], but also [9 and 10]), while for $m > 1$ we prove it with techniques similar to those used by Hörmander in [3]: our method, indeed, uses $L^2$-estimates for solutions of the Cauchy-Riemann equation, together with the
characterization of the space \( \mathcal{S}'(\Gamma) \), provided in [9], and could be suitably modified to yield the solution for a similar problem in \( \mathcal{D}'(\Gamma) \), the space of finite-order distributions with support in \( \Gamma \).

1. According to Ehrenpreis [2], we say that a compactly supported distribution \( S \in \mathcal{E}'(\mathbb{R}^n) \) is invertible if, with \( \hat{S} \) denoting the Fourier transform of \( S \), there are constants \( A_1, A_2, A_3 > 0 \) such that for every \( \xi \in \mathbb{R}^n \) there exists \( \eta \in \mathbb{R}^n \) satisfying

\[
|\xi - \eta| \leq A_1 \log(2 + |\xi|) \quad \text{and} \quad |\hat{S}(\eta)| \geq (A_2 + |\xi|)^{-A_3}.
\]

It is well known that a distribution \( S \) in \( \mathcal{E}'(\mathbb{R}^n) \) is invertible if and only if \( S * \mathcal{D}'(\mathbb{R}^n) = \mathcal{D}'(\mathbb{R}^n) \).

If \( K \) is a compact convex set in \( \mathbb{R}^n \), \( H_K(\xi) = \sup_{x \in K} \langle x, \xi \rangle \) will denote its supporting function; denote by \( \mathcal{H} \) the set of all supporting functions of compact convex sets (including the empty set, whose supporting function is \( -\infty \)): then, if \( S \in \mathcal{E}'(\mathbb{R}^n) \), \( \mathcal{H}(S) \) denotes the set of all \( h \in \mathcal{H} \) such that there is a sequence \( \xi \nu \to \infty \) in \( \mathbb{R}^n \) with

\[
L_S(z, \xi \nu) = \frac{\log |\hat{S}(\xi \nu + z \log |\xi \nu|)|}{\log |\xi \nu|}
\]

converging to a plurisubharmonic function with supporting function \( h \) [4].

Let us now restrict our attention to the case \( n = 1 \); let \( S \in \mathcal{E}'(\mathbb{R}) \) be a distribution) with compact support, \( \text{supp}(S) \subseteq [0, +\infty) \); let \( \mathcal{D}'([0, +\infty)) \) be the space of distributions with supports contained in \( [0, +\infty) \) and \( \mathcal{D}'_+ \) the space of distributions with supports contained in \( [a, +\infty) \), for some \( a \in \mathbb{R} \). We have:

**Theorem 1.** Let \( S \in \mathcal{E}'(\mathbb{R}), \text{supp}(S) \subseteq [0, +\infty) \). Then \( S * \mathcal{D}'([0, +\infty)) = \mathcal{D}'([0, +\infty)) \) if and only if \( S \) is invertible and

for all \( j \) in \( \mathbb{Z} \), for all \( a \) in \( \mathbb{R} \), there exists \( b \) in \( \mathbb{R} \) such that for all \( h \in \mathcal{H}(S) \) and for all \( x \in \mathbb{R} \)

\[
(1) \quad h(\xi) + x \cdot \xi \leq H_{[j, a]}(\xi), \quad \text{for all } \xi, \text{ implies } x \in [j, b].
\]

**Proof.** In view of Corollary 1 of [8], we only have to show that (1) is equivalent to

\[
(2) \quad \text{for all } j \in \mathbb{Z} \text{ and all } u \in \mathcal{D}'_+, \text{ sing supp}(S * u) \subseteq [j, +\infty) \text{ implies sing supp}(u) \subseteq [j, +\infty).}
\]

We first notice that condition (2) is equivalent to

\[
(3) \quad \text{for all } j \in \mathbb{Z} \text{ and all } u \in \mathcal{E}', \text{ sing supp}(S * u) \subseteq [j, +\infty) \text{ implies sing supp}(u) \subseteq [j, +\infty).}
\]

It is clear that (2) implies (3), as \( \mathcal{E}' \subseteq \mathcal{D}'_+ \). On the other hand, let (3) hold, \( u \in \mathcal{D}'_+ \) and \( \text{sing supp}(S * u) \subseteq [j, +\infty) \) for some \( j \). Let \( \alpha = \inf\{x: x \in \text{supp}(u)\} \); if \( \alpha \geq j \) we do not need to prove anything, otherwise take \( \phi \in C_0^\infty(\mathbb{R}) \), \( \phi \equiv 1 \) on \( [\alpha, j] \), \( \phi \geq 0 \), and define

\[
T_1 = \phi \cdot u \text{ and } T_2 = (1 - \phi) \cdot u,
\]

so that \( u = T_1 + T_2 \). Notice that \( \text{supp}(T_2) \subseteq [j, +\infty) \), hence \( \text{sing supp}(T_2) \subseteq [j, +\infty) \). Now

\[
S * T_1 = S * u - S * T_2;
\]
notice that both $S * u$ and $S * T_2$ are $C^\infty$ in $(-\infty,j)$, because $\text{supp}(T_2) \subseteq [j, +\infty)$ and $\text{supp}(S) \subseteq [0, +\infty)$; therefore

$$\text{sing supp}(S * T_1) \subseteq [j, +\infty).$$

Now $T_1$ is of compact support, and hence (3) implies that $\text{sing supp}(T_1) \subseteq [j, +\infty)$. Finally, as $u = T_1 + T_2$, we get that $\text{sing supp}(u) \subseteq [j, +\infty)$. The equivalence of (3) and (1) is now an immediate consequence of Theorem 16.3.13 of [4].

Condition (1) or, equivalently, condition (3), enables us to give some classes of examples of surjective convolutors onto $\mathcal{D}'([0, +\infty))$; none of them were given in [8].

**EXAMPLE 1.** Let $S$ be a compactly supported invertible distribution with $\text{sing supp}(S') = \{0\}$; then we can either use Corollary 16.3.15 of [4], or remark that, in this case, the set $\mathcal{K}(S)$ contains only one function, namely the supporting function of the convex hull of the singular support of $S$; in both ways one sees that (1) is satisfied, and therefore $S*$ is a surjection.

**EXAMPLE 2.** Suppose $\hat{S}$ is an exponential polynomial (this means that $S$ is a distribution with finite support) such that $0 \in \text{sing supp}(S)$. Then it is well known that $S$ is invertible and Corollary 16.3.18 of [4] implies that (1) is satisfied.

**EXAMPLE 3.** Let $\mu, \nu \in \mathcal{E}'(\mathbb{R})$; suppose that $\mu$ contains an atom, $\text{sing supp}(\mu) \cap \text{sing supp}(\nu) = \emptyset$, and that $\mathcal{K}(\mu)$ and $\mathcal{K}(\nu)$ only contain one function. Then if $S = \nu + P(D)\mu$, it is shown in [1] that $S$ is invertible and in [4] that $\mathcal{K}(S)$ contains only one function, and therefore $S$ induces, by convolution, a surjection onto $\mathcal{D}'([0, +\infty))$. In particular, one could take $\nu$ to have a singular support constituted only by isolated points; see Corollary 16.3.18 in [4].

**REMARK.** In case $\mathcal{K}(S)$ contains only one function, (1) implies that

$$\text{sing supp}(S * u) \subseteq [j, a] \implies a - j > a,$$

where the convex hull of $\text{sing supp}(S)$ is $[0, a]$, and in (1), $b$ can be taken to be $a - \alpha$.

We now turn to the case of several variables. Let $\Gamma \subset \mathbb{R}^n$ be a closed convex cone whose vertex is the origin, and which does not contain any straight line. We denote by $\mathcal{D}'(\Gamma)$ the space of distributions supported in $\Gamma$, and by $\mathcal{D}_c^\infty(\Gamma)$ the space of distributions with support contained in some translate of $\Gamma$. It is well known [5] that if $S \in \mathcal{E}'$ has support contained in $\Gamma$, then $S * \mathcal{D}'(\Gamma) \subseteq \mathcal{D}'(\Gamma)$. A complete characterization of those $S \in \mathcal{E}'$ for which $S * \mathcal{D}'(\Gamma) = \mathcal{D}'(\Gamma)$ has been given by Hörmander [4], in Theorems 16.7.3, 16.7.4; in particular the condition asks that for every closed cone $\Gamma_1 \subset \Gamma$, there are positive constants $C, M, A$ such that

$$(4) \quad |1/\hat{S}(z)| < C(1 + |z|)^M \exp(A|\text{Im } z|)$$

if $\text{Im } z \in \Gamma_1$ and $|\text{Im } z| > C \log(2 + |z|)$.

At first sight, it does not seem obvious to verify that every invertible distribution $S \in \mathcal{E}'(\mathbb{R})$ for which $\mathcal{K}(S)$ consists only of one function satisfies (4), but this is indeed an immediate consequence of Theorem 1 and Hörmander’s results.

The result of [8], Theorem 1, can be extended in an obvious way to the case of distributions supported in cones; the proof uses the same arguments as in [8], and the Lion-Titchmarsh theorems for supports of convolutions, [5]: for this reason it
does not seem necessary to repeat it here; in this case the surjectivity condition is that $S$ must be invertible, and that

$$\text{for all } a \in \mathbb{R}^n \text{ and all } u \in \mathcal{D}'_\Gamma, \text{sing supp}(S \ast u) \subseteq \Gamma + \{a\} \text{ implies sing supp}(u) \subseteq \Gamma + \{a\}.$$  

However, it is easy to show that, in (5), one cannot replace $\mathcal{D}'_\Gamma$ with $\mathcal{E}'$. To prove it, one simply considers any partial differential operator $P(D)$ which is not hyperbolic with respect to $\Gamma$. Then it is obvious that condition (5) with $\mathcal{D}'_\Gamma$ replaced by $\mathcal{E}'$ is satisfied, while (5) itself is not.

2. In this section $\Gamma$ will be an open convex cone, with vertex at the origin, and such that it does not contain any straight line; denote by $\Gamma^*$ its dual cone $\Gamma^* = \{\xi \in \mathbb{R}^n : \langle \xi, x \rangle \geq 0 \text{ for all } x \in \Gamma\}$, and let $S'(\Gamma^*)$ be the space of tempered distributions with supports contained in the closed cone $\Gamma^*$. Consider $S_1, \ldots, S_m \in S'(\Gamma^*)$: we look for necessary and sufficient conditions on $S_1, \ldots, S_m$ in order to ensure the existence of $E_1, \ldots, E_m \in S'(\Gamma^*)$ such that

$$\text{(6) } S_1 * E_1 + \cdots + S_m * E_m = \delta.$$  

It is well known [9] that the Laplace transform is an isomorphism between $S'(\Gamma^*)$ and the space $H(\Gamma)$ of functions $f$ holomorphic in $\mathbb{R}^n + i\Gamma$ for which there are positive constants $\alpha, \beta, M$ such that

$$|f(z)| \leq M(1 + |z|^2)^\alpha (1 + \Delta^{-\beta}(|\text{Im} z|)),$$  

for all $z \in \mathbb{R}^n + i\Gamma$, where $\Delta(y) = \text{dist}(y, \partial \Gamma)$.

**Theorem 2.** Given $S_1, \ldots, S_m \in S'(\Gamma^*)$, a necessary and sufficient condition for the existence of $E_1, \ldots, E_m \in S'(\Gamma^*)$ such that (6) holds, is that there exists positive constants $C, \gamma, \delta$ such that, for all $z \in \mathbb{R}^n + i\Gamma$,

$$\left| S_1(z) \right| + \cdots + \left| S_m(z) \right| \geq C(1 + |z|^2)^{-\gamma}(1 + \Delta^{-\delta}(|\text{Im} z|))^{-1}.$$  

**Proof.** By taking the Laplace transform of (6), the necessity of (7) is obvious. We only sketch the proof of its sufficiency, since it follows the well-known lines of [3]. First notice that even if $H(\Gamma)$ is not an $A_p$ space in the sense of [3], it does satisfy the properties which are necessary to apply the $\bar{\partial}$-techniques; indeed, since $\Gamma$ is convex, the function $\log(1 + \Delta^{-\beta}(|\text{Im} z|))$ is plurisubharmonic; on the other hand, using the Laplace transform, one immediately sees that if $f \in H(\Gamma)$ its derivatives also belong to $H(\Gamma)$. In order to conclude, as in [3], with the construction of a suitable Koszul complex, it is sufficient to show that if an analytic function $f$ satisfies an $L^2$-estimate of the form

$$\int_{\mathbb{R}^n + i\Gamma} |f(z)|^2(1 + |z|^2)^{-\gamma}(1 + \Delta^{-\delta}(|\text{Im} z|))^{-1} d\lambda < +\infty,$$

where $d\lambda$ is the Lebesgue measure, then $f \in H(\Gamma)$. To prove this, it is sufficient to apply the mean value theorem to $f$ on a ball centered at $z$ and of radius $\Delta(|\text{Im} z|)/2$, and then to use the Cauchy-Schwarz inequality.

**References**


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