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ONE EXPRESSION FOR THE SOLUTIONS OF SECOND ORDER DIFFERENCE EQUATIONS

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ABSTRACT. Explicit formulas for the general solutions of homogeneous and nonhomogeneous second order linear difference equations with arbitrarily varying coefficients are obtained.

1. Introduction. In this note we give some explicit formulas for the solutions of second order difference equations. The obtained expressions can be used both in theoretical investigations and numerical methods. Note that similar results are not known for differential equations. To obtain these formulas we shall introduce some special sum operators.

We shall use the following notation.

(a) N; the set of positive integers,

(b) R; the set of real numbers, and

(c) x_n; the value of function x at the point $n \in N$.

Let $x\colon N \to R$. We define

$$\sum\nolimits_{(n,k)} x_i := \sum_{\substack{d_1,\dots,d_n=0 \\ d_i d_{i+1}=0 \\ i=1,\dots,n-1}}^{1} \prod_{i=1}^{n} x_{i+k}^{d_i} \quad \text{for } n > 1, k \geq 0, \tag{i}$$

and

$$\sum\nolimits_{(1,k)} x_i := \sum_{d_1=0}^{1} x_{1+k}^{d_1}. \tag{ii}$$

We assume

$$\sum\nolimits_{(0,k)} x_i = 1, \quad \sum\nolimits_{(-1,k)} x_i = 1, \quad \sum\nolimits_{(-2,k)} x_i = 0, \tag{iii}$$

$$\sum_{i=k}^{k-t} x_i = 0, \quad \prod_{i=k}^{k-t} x_i = 1 \quad \text{for all } k, t \in N, \tag{iv}$$

and

$$0^0 = 1, \quad 0^1 = 0. \tag{v}$$

The forward difference operator is defined as usual;

$$\Delta x_n = x_{n+1} - x_n, \qquad \Delta^2 x_n = \Delta(\Delta x_n).$$

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2. Main results.

THEOREM 1. *Let $q\colon N \to R$. Then every solution of*

$$x_{n+2} = x_{n+1} + q_n x_n, \qquad n \in N, \tag{1}$$

can be represented by the formula

$$x_n = q_1 x_1 \sum\nolimits_{(n-4,2)} q_i + x_2 \sum\nolimits_{(n-3,1)} q_i, \qquad n \geq 2, \tag{2}$$

where x_1, x_2 are arbitrary real constants.

PROOF. The proof of this theorem follows by induction. We see, by (iii), that (2) becomes an identity for $n = 2$. Furthermore it is not difficult to check (2) for $n = 3, 4, 5$ and compare x_3, x_4, x_5 with the same x_i calculated in succession from (1).

Let (2) hold for $n = m$ and $n = m + 1$. We prove that it remains true for $n = m + 2$. From (1) we get

$$\begin{aligned}
x_{m+2} &= x_{m+1} + q_m x_m \\
&= q_1 x_1 \sum\nolimits_{(m-3,2)} q_i + x_2 \sum\nolimits_{(m-2,1)} q_i \\
&\quad + q_m \left[q_1 x_1 \sum\nolimits_{(m-4,2)} q_i + x_2 \sum\nolimits_{(m-3,1)} q_i \right] \\
&= q_1 x_1 \left[\sum\nolimits_{(m-3,2)} q_i + q_m \sum\nolimits_{(m-4,2)} q_i \right] \\
&\quad + x_2 \left[\sum\nolimits_{(m-2,1)} q_i + q_m \sum\nolimits_{(m-3,1)} q_i \right] \\
&= q_1 x_1 \left[q^0_{m-2+2} \sum\nolimits_{(m-3,2)} q_i + q^1_{m-2+2} q^0_{m-3+2} \sum\nolimits_{(m-4,2)} q_i \right] \\
&\quad + x_2 \left[q^0_{m-1+1} \sum\nolimits_{(m-2,1)} q_i + q^1_{m-1+1} q^0_{m-2+1} \sum\nolimits_{(m-3,1)} q_i \right] \\
&= q_1 x_1 \sum\nolimits_{(m-2,2)} q_i + x_2 \sum\nolimits_{(m-1,1)} q_i. \quad \text{Q.E.D.}
\end{aligned}$$

We shall now consider the equation

$$x_{n+2} = a x_{n+1} + q_n x_n, \qquad n \in N, \tag{3}$$

where $q\colon N \to R$, a is any fixed real constant.

We know that for the case $a = 0$ the solution of (3) is

$$x_n = \begin{cases} x_1 \prod\limits_{j=1}^{k} q_{2j-1} & \text{for } n = 2k+1,\ k = 0, 1, \ldots, \\ x_2 \prod\limits_{j=1}^{k} q_{2j} & \text{for } n = 2k+2,\ k = 0, 1, \ldots. \end{cases}$$

Now let $a \neq 0$. After dividing (3) by a^{n+2}, it can be put into the form

$$y_{n+2} = y_{n+1} + \frac{q_n}{a^2} y_n,$$

where $y_n = x_n/a^n$, $n \in N$. Hence by Theorem 1 we obtain

$$y_n = a^{-2} q_1 y_1 \sum\nolimits_{(n-4,2)} p_i + y_2 \sum\nolimits_{(n-3,1)} p_i,$$

where $p_n = a^{-2} q_n$, $n \in N$. Therefore

$$x_n = a^{n-3} q_1 x_1 \sum\nolimits_{(n-4,2)} p_i + a^{n-2} x_2 \sum\nolimits_{(n-3,1)} p_i, \quad n \geq 2, \tag{4}$$

with p_n defined above and x_1, x_2 arbitrary constants. In explicit form expression (4) is equivalent to

$$\begin{aligned} x_n = x_1 q_1 & \sum_{\substack{d_1,\dots,d_{n-4}=0 \\ d_i d_{i+1}=0 \\ i=1,\dots,n-5}}^{1} \left[\prod_{i=1}^{n-4} q_{i+2}^{d_i} \right] a^{(n-3-2\sum_{i=1}^{n-4} d_i)} \\ + x_2 & \sum_{\substack{d_1,\dots,d_{n-3}=0 \\ d_i d_{i+1}=0 \\ i=1,\dots,n-4}}^{1} \left[\prod_{i=1}^{n-3} q_{i+1}^{d_i} \right] a^{(n-2-2\sum_{i=1}^{n-3} d_i)}, \quad n \geq 5. \end{aligned}$$

Equation (3) is connected with

$$\Delta^2 x_n = r_n x_n, \quad n \in N, \tag{5}$$

because (5) can be written in the form

$$x_{n+2} = 2x_{n+1} + (1 + r_n) x_n, \quad n \in N.$$

Having obtained the solution (4) of (3), we get the general solution of (5). Specifically,

$$x_n = 2^{n-3}(1 + r_1) x_1 \sum\nolimits_{(n-4,2)} z_i + 2^{n-2} x_2 \sum\nolimits_{(n-3,1)} z_i, \quad n \geq 2,$$

where $z_n = \frac{1}{4}(1 + r_n)$, $n \in N$, and x_1, x_2 are arbitrary.

We aim our attention to the equation

$$x_{n+2} = a_n x_{n+1} + q_n x_n, \quad n \in N, \tag{6}$$

where $a \colon N \to R \backslash \{0\}$, $q \colon N \to R$. We shall set $b_{n+2} = a_n$, $n \in N$, $b_1 = b_2 = 1$. Dividing (6) by $\prod_{j=1}^{n+2} b_j$ and putting

$$y_n = x_n \prod_{j=1}^{n} \frac{1}{b_j}, \quad n \in N,$$

we have

$$y_{n+2} = y_{n+1} + \frac{q_n}{b_{n+2} b_{n+1}} y_n, \qquad n \in N.$$

Consequently by Theorem 1

$$y_n = \frac{q_1}{b_3} y_1 \sum\nolimits_{(n-4,2)} z_i + y_2 \sum\nolimits_{(n-3,1)} z_i,$$

where

$$z_n = \frac{q_n}{b_{n+2} b_{n+1}}, \qquad n \in N.$$

Therefore the general solution of (6) is

$$x_n = x_1 q_1 \left[\prod_{j=2}^{n-2} a_j\right] \sum\nolimits_{(n-4,2)} z_i + x_2 \left[\prod_{j=1}^{n-2} a_j\right] \sum\nolimits_{(n-3,1)} z_i, \qquad n \geq 2,$$

where $z_n = q_n/a_n a_{n-1}$, $n \geq 2$, $z_1 = q_1/a_1$, and x_1, x_2 are arbitrary constants.

Having solved (6) we can obtain the general solution of a linear homogeneous second order equation

$$a_n x_{n+2} + b_n x_{n+1} + c_n x_n = 0, \qquad n \in N, \tag{7}$$

where $a, b\colon N \to R\backslash\{0\}$, and $c\colon N \to R$. This solution is

$$\begin{aligned} x_n = {} & (-1)^n x_1 \frac{c_1}{a_1} \left[\prod_{j=2}^{n-2} \frac{b_j}{a_j}\right] \sum\nolimits_{(n-4,2)} z_i \\ & + (-1)^n x_2 \left[\prod_{j=1}^{n-2} \frac{b_j}{a_j}\right] \sum\nolimits_{(n-3,1)} z_i, \qquad n \geq 2, \end{aligned} \tag{8}$$

where x_1, x_2 are arbitrary constants and

$$z_n = -\frac{c_n a_{n-1}}{b_n b_{n-1}} \text{ for } n > 1.$$

Considering the linear nonhomogeneous equation we have the following result.

THEOREM 2. *Let $b, c\colon N \to R\backslash\{0\}$, $r\colon N \to R$,*

$$\sum\nolimits_{(n,1)} z_i \neq 0 \quad \textit{for } n \in N, \tag{9}$$

where

$$z_n = \frac{-c_n}{b_n b_{n-1}}, \qquad n \geq 2. \tag{10}$$

Then the general solution of

$$x_{n+2} + b_n x_{n+1} + c_n x_n = r_n, \qquad n \in N, \tag{11}$$

is

$$\begin{aligned} x_n = {} & (-1)^n x_2 \left[\prod_{j=1}^{n-2} b_j\right] \sum\nolimits_{(n-3,1)} z_i \\ & + (-1)^{n+1} x_1 c_1 \left[\prod_{j=1}^{n-2} b_j\right] \left[\sum\nolimits_{(n-3,1)} z_i\right] \sum_{k=1}^{n-2} \left[\prod_{j=2}^{k+1} \frac{1}{c_j}\right] \frac{1}{y_{k+1} y_{k+2}} \\ & + (-1)^n \left[\prod_{j=1}^{n-2} b_j\right] \left[\sum\nolimits_{(n-3,1)} z_i\right] \sum_{k=1}^{n-2} \left[\prod_{j=2}^{k+1} \frac{1}{c_j}\right] \frac{1}{y_{k+1} y_{k+2}} \sum_{j=1}^{k} r_j y_{j+1} \end{aligned}$$

for $n \geq 2$, where x_1, x_2 are arbitrary constants, and

$$y_n = (-1)^n \left[\prod_{j=1}^{n-2} \frac{b_j}{c_{j+1}}\right] \sum\nolimits_{(n-3,1)} z_i, \qquad n \geq 2.$$

PROOF. In order to obtain (12), we consider the equation

$$c_{n+1}y_{n+2} + b_n y_{n+1} + y_n = 0, \qquad n \in N. \tag{13}$$

The general solution of (13) is, by (8),

$$y_n = (-1)^n \left[\prod_{j=1}^{n-2} \frac{b_j}{c_{j+1}}\right] \left[\frac{1}{b_1} y_1 \sum\nolimits_{(n-4,2)} z_i + y_2 \sum\nolimits_{(n-3,1)} z_i\right], \tag{14}$$

where z_n is defined by (10). Let y_n satisfy the initial conditions $y_1 = 0, y_2 = 1$. As a result, we get from (14)

$$y_n = (-1)^n \left[\prod_{j=1}^{n-2} \frac{b_j}{c_{j+1}}\right] \sum\nolimits_{(n-3,1)} z_i, \qquad n \geq 2. \tag{15}$$

Under the hypothesis (9), the solution (15) vanishes at no point n, $n \geq 2$.

On multiplying (11) by y_{n+1} and writing (13) in the form $b_n y_{n+1} = -y_n - c_{n+1}y_{n+2}$ we have $(y_{n+1}x_{n+2} - c_{n+1}y_{n+2}x_{n+1}) - (y_n x_{n+1} - c_n y_{n+1} x_n) = r_n y_{n+1}$. On summing the above equation with respect to n one gets

$$y_{n+1}x_{n+2} - c_{n+1}y_{n+2}x_{n+1} = y_1 x_2 - c_1 y_2 x_1 + \sum_{j=1}^{n} r_j y_{j+1}, \qquad n \in N. \tag{16}$$

Setting $y_1 = 0$ and $y_2 = 1$, dividing (16) by $y_{n+1}y_{n+2}\prod_{j=1}^{n} c_{j+1}$, and summing next with respect to n, we have

$$\begin{aligned} \frac{x_{n+2}}{y_{n+2}} \left[\prod_{j=1}^{n} \frac{1}{c_{j+1}}\right] - x_2 = & - c_1 x_1 \sum_{k=1}^{n} \left[\prod_{j=1}^{k} \frac{1}{c_{j+1}}\right] \frac{1}{y_{k+1}y_{k+2}} \\ & + \sum_{k=1}^{n} \left[\prod_{j=1}^{k} \frac{1}{c_{j+1}}\right] \frac{1}{y_{k+1}y_{k+2}} \sum_{j=1}^{k} r_j y_{j+1}. \end{aligned}$$

Thus

$$\begin{aligned} x_{n+2} = {} & x_2 y_{n+2} \prod_{j=1}^{n} c_{j+1} - c_1 x_1 y_{n+2} \left[\prod_{j=1}^{n} c_{j+1}\right] \sum_{k=1}^{n} \left[\prod_{j=1}^{k+1} \frac{1}{c_j}\right] \frac{1}{y_{k+1}y_{k+2}} \\ & + y_{n+2} \left[\prod_{j=1}^{n} c_{j+1}\right] \sum_{k=1}^{n} \left[\prod_{j=1}^{k+1} \frac{1}{c_j}\right] \frac{1}{y_{k+1}y_{k+2}} \sum_{j=1}^{k} r_j y_{j+1}, \qquad n \in N. \end{aligned}$$

Hence, by (15) we infer that (12) holds for $n > 2$. Furthermore, by (iii) and (iv), (12) is also valid for $n = 2$. Q.E.D.

Note that equation (11) can possibly be solved even in the case when (9) is not fulfilled, that is, even if the y_n given by (15) vanished for some n. The y_n given by (15) is one particular solution of (13). The general solution is given by (14). If the y_n given by (15) vanished for some $n \geq 2$, then any other particular solution obtained from (14) which did not become zero for any $n \in N$ could be used. This solution rather than (15) should be substituted in the following equality:

$$x_{n+2} = y_{n+2}\left[\prod_{j=2}^{n+1} c_j\right]\left\{\frac{x_2}{y_2} + \sum_{k=1}^{n}\left[\frac{1}{y_{k+1}y_{k+2}}\left[\prod_{j=2}^{k+1}\frac{1}{c_j}\right]\right.\right.$$
$$\left.\left.\cdot\left(y_1x_2 - c_1y_2x_1 + \sum_{j=1}^{k} r_jy_{j+1}\right)\right]\right\}, \qquad n \in N.$$

However, the obtained expression for the solution of (11) will be more complicated in any case.

Finally, we see that the equation

$$a_nx_{n+2} + b_nx_{n+1} + c_nx_n = r_n, \tag{17}$$

where $a, b, c\colon N \to R\backslash\{0\}$, $r\colon N \to R$, can be solved by applying Theorem 2.

On dividing (17) by a_n, we get

$$x_{n+2} + \frac{b_n}{a_n}x_{n+1} + \frac{c_n}{a_n}x_n = \frac{r_n}{a_n}$$

which is of the type considered in Theorem 2. Therefore, to get its solution we need a nonvanishing solution of the equation

$$\frac{c_{n+1}}{a_{n+1}}y_{n+2} + \frac{b_n}{a_n}y_{n+1} + y_n = 0.$$

If for instance $\sum_{(n,1)} z_i \neq 0$ for all $n \in N$, where $z_n = -c_na_{n-1}/b_nb_{n-1}$, we can write the general solution of (17) in the form

$$x_n = (-1)^n\left[\prod_{j=1}^{n-2}\frac{b_j}{a_j}\right]\left[\sum\nolimits_{(n-3,1)} z_i\right]$$
$$\cdot\left\{x_2 - \frac{c_1}{a_1}x_1\sum_{k=1}^{n-2}\left[\prod_{j=2}^{k+1}\frac{a_j}{c_j}\right]\frac{1}{y_{k+1}y_{k+2}}\right.$$
$$\left.+\sum_{k=1}^{n-2}\left[\prod_{j=2}^{k+1}\frac{a_j}{c_j}\right]\frac{1}{y_{k+1}y_{k+2}}\sum_{j=1}^{k}\frac{r_j}{a_j}y_{j+1}\right\}, \qquad n > 2,$$

where x_1, x_2 are arbitrary constants, z_n is defined above, and

$$y_n = (-1)^n\frac{a_{n-1}}{a_1}\left[\prod_{j=1}^{n-2}\frac{b_j}{c_{j+1}}\right]\sum\nolimits_{(n-3,1)} z_i, \qquad n \geq 2,\ y_1 = 0.$$

3. Nonlinear examples. Various equations can be made linear by a change of variable. In particular, such a type is Riccati's equation

$$a_nx_{n+1}x_n + b_nx_n + c_n = 0, \tag{18}$$

where $a, b, c\colon N \to R\backslash\{0\}$. Make the change of variable $x_n = y_{n+1}/y_n$, with $y_n \neq 0$ for all $n \in N$. Then (18) becomes

$$a_ny_{n+2} + b_ny_{n+1} + c_ny_n = 0.$$

Hence, by (8)

$$x_n = -\frac{b_{n-1}}{a_{n-1}} \left\{ \frac{c_1 \sum_{(n-3,2)} z_i + b_1 x_1 \sum_{(n-2,1)} z_i}{c_1 \sum_{(n-4,2)} z_i + b_1 x_1 \sum_{(n-3,1)} z_i} \right\}, \qquad n \geq 2,$$

where $z_n = -c_n a_{n-1}/b_n b_{n-1}$, $n > 1$, x_1 is an arbitrary constant.

For $y_n \neq 0$ we need

$$c_1 \sum\nolimits_{(n-3,2)} z_i + b_1 x_1 \sum\nolimits_{(n-2,1)} z_i \neq 0 \quad \text{for all } n \in N.$$

The second example treats the equation

$$a_n x_{n+1} x_n + b_n x_{n+1} + c_n = 0. \tag{19}$$

Here, by putting $x_n = y_n/y_{n+1}$, $y_n \neq 0$ on N, we get

$$c_n y_{n+2} + b_n y_{n+1} + a_n y_n = 0.$$

Supposing $a, b, c\colon N \to R\backslash\{0\}$, and

$$a_1 x_1 \sum\nolimits_{(n-4,2)} z_i + b_1 \sum\nolimits_{(n-3,1)} z_i \neq 0 \quad \text{for all } n \in N,$$

where $z_n = -a_n c_{n-1}/b_n b_{n-1}$, $n > 1$, we obtain, by (8), the expression of the solution of the considered equation. That is, the solution x_n of (19) is given by

$$x_n = -\frac{c_{n-1}}{b_{n-1}} \left\{ \frac{a_1 x_1 \sum_{(n-4,2)} z_i + b_1 \sum_{(n-3,1)} z_i}{a_1 x_1 \sum_{(n-3,2)} z_i + b_1 \sum_{(n-2,1)} z_i} \right\}, \qquad n \geq 2.$$

4. Note that the operators introduced in this paper can be used with some modification for recurrence equations of higher order.

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