ONE EXPRESSION FOR THE SOLUTIONS
OF SECOND ORDER DIFFERENCE EQUATIONS

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ABSTRACT. Explicit formulas for the general solutions of homogeneous and
nonhomogeneous second order linear difference equations with arbitrarily vary-
ing coefficients are obtained.

1. Introduction. In this note we give some explicit formulas for the solutions
of second order difference equations. The obtained expressions can be used both
in theoretical investigations and numerical methods. Note that similar results are
not known for differential equations. To obtain these formulas we shall introduce
some special sum operators.

We shall use the following notation.
(a) \( N \); the set of positive integers,
(b) \( R \); the set of real numbers, and
(c) \( x_n \); the value of function \( x \) at the point \( n \in N \).

Let \( x : N \to R \). We define

\[
(i) \quad \sum_{(n,k)} x_i := \sum_{d_1, \ldots, d_n=0}^{d_i \geq 0} \prod_{i=1}^{n} x_{i+k}^{d_i}
\]

for \( n > 1, k \geq 0 \),

and

\[
(ii) \quad \sum_{(1,k)} x_i := \sum_{d_1=0}^{1} x_{1+k}^{d_1}.
\]

We assume

\[
(iii) \quad \sum_{(0,k)} x_i = 1, \quad \sum_{(-1,k)} x_i = 1, \quad \sum_{(-2,k)} x_i = 0,
\]

\[
(iv) \quad \sum_{i=k}^{k-t} x_i = 0, \quad \prod_{i=k}^{k-t} x_i = 1 \quad \text{for all } k, t \in N,
\]

and

\[
(v) \quad 0^0 = 1, \quad 0^1 = 0.
\]

The forward difference operator is defined as usual;

\[
\Delta x_n = x_{n+1} - x_n, \quad \Delta^2 x_n = \Delta(\Delta x_n).
\]

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2. Main results.

THEOREM 1. Let \( q: \mathbb{N} \rightarrow \mathbb{R} \). Then every solution of
\[
(1) \quad x_{n+2} = x_{n+1} + q_n x_n, \quad n \in \mathbb{N},
\]
can be represented by the formula
\[
(2) \quad x_n = q_1 x_1 \sum_{(n-4,2)} q_i + x_2 \sum_{(n-3,1)} q_i, \quad n \geq 2,
\]
where \( x_1, x_2 \) are arbitrary real constants.

PROOF. The proof of this theorem follows by induction. We see, by (iii), that (2) becomes an identity for \( n = 2 \). Furthermore it is not difficult to check (2) for \( n = 3, 4, 5 \) and compare \( x_3, x_4, x_5 \) with the same \( x_i \) calculated in succession from (1).

Let (2) hold for \( n = m \) and \( n = m + 1 \). We prove that it remains true for \( n = m + 2 \). From (1) we get
\[
x_{m+2} = x_{m+1} + q_m x_m
\]
\[
= q_1 x_1 \sum_{(m-3,2)} q_i + x_2 \sum_{(m-3,1)} q_i
\]
\[
+ q_m \left[ q_1 x_1 \sum_{(m-4,2)} q_i + x_2 \sum_{(m-3,1)} q_i \right]
\]
\[
= q_1 x_1 \left[ \sum_{(m-3,2)} q_i + q_m \sum_{(m-4,2)} q_i \right]
\]
\[
+ x_2 \left[ \sum_{(m-2,1)} q_i + q_m \sum_{(m-3,1)} q_i \right]
\]
\[
= q_1 x_1 \left[ \sum_{(m-2,2)} q_i + x_2 \sum_{(m-1,1)} q_i \right]. \quad \text{Q.E.D.}
\]

We shall now consider the equation
\[
(3) \quad x_{n+2} = a x_{n+1} + q_n x_n, \quad n \in \mathbb{N},
\]
where \( q: \mathbb{N} \rightarrow \mathbb{R}, a \) is any fixed real constant.

We know that for the case \( a = 0 \) the solution of (3) is
\[
x_n = \begin{cases} 
  x_1 \prod_{j=1}^{k} q_{2j-1} & \text{for } n = 2k + 1, \ k = 0, 1, \ldots, \\
  x_2 \prod_{j=1}^{k} q_{2j} & \text{for } n = 2k + 2, \ k = 0, 1, \ldots.
\end{cases}
\]

Now let \( a \neq 0 \). After dividing (3) by \( a^{n+2} \), it can be put into the form
\[
y_{n+2} = y_{n+1} + \frac{q_n}{a^2} y_n,
\]
where \( y_n = x_n/a^n, \) \( n \in N. \) Hence by Theorem 1 we obtain

\[
y_n = a^{-2} q_1 y_1 \sum_{(n-4,2)} p_i + y_2 \sum_{(n-3,1)} p_i,
\]

where \( p_n = a^{-2} q_n, \) \( n \in N. \) Therefore

\[
x_n = a^{n-3} q_1 x_1 \sum_{(n-4,2)} p_i + a^{n-2} x_2 \sum_{(n-3,1)} p_i, \quad n \geq 2,
\]

with \( p_n \) defined above and \( x_1, x_2 \) arbitrary constants. In explicit form expression (4) is equivalent to

\[
x_n = x_1 q_1 \sum_{d_1, \ldots, d_{n-4}=0}^{1} \left[ \prod_{i=1}^{n-4} q_{i+2}^{d_i} \right] a^{n-3-2 \sum_{i=1}^{n-4} d_i} + x_2 \sum_{d_1, \ldots, d_{n-3}=0}^{1} \left[ \prod_{i=1}^{n-3} q_{i+1}^{d_i} \right] a^{n-2-2 \sum_{i=1}^{n-3} d_i}, \quad n \geq 5.
\]

Equation (3) is connected with

\[
\Delta^2 x_n = r_n x_n, \quad n \in N,
\]

because (5) can be written in the form

\[
x_{n+2} = 2x_{n+1} + (1 + r_n)x_n, \quad n \in N.
\]

Having obtained the solution (4) of (3), we get the general solution of (5). Specifically,

\[
x_n = 2^{n-3}(1 + r_1)x_1 \sum_{(n-4,2)} z_i + 2^{n-2} x_2 \sum_{(n-3,1)} z_i, \quad n \geq 2,
\]

where \( z_n = \frac{1}{4}(1 + r_n), \) \( n \in N, \) and \( x_1, x_2 \) are arbitrary.

We aim our attention to the equation

\[
x_{n+2} = a_n x_{n+1} + q_n x_n, \quad n \in N,
\]

where \( a: N \rightarrow R \backslash \{0\}, \) \( q: N \rightarrow R. \) We shall set \( b_{n+2} = a_n, \) \( n \in N, \) \( b_1 = b_2 = 1. \) Dividing (6) by \( \prod_{j=1}^{n+2} b_j \) and putting

\[
y_n = x_n \prod_{j=1}^{n} \frac{1}{b_j}, \quad n \in N,
\]

we have

\[
y_{n+2} = y_{n+1} + \frac{q_n}{b_{n+2} b_{n+1}} y_n, \quad n \in N.
\]

Consequently by Theorem 1

\[
y_n = \frac{q_1}{b_3} y_1 \sum_{(n-4,2)} z_i + y_2 \sum_{(n-3,1)} z_i,
\]

where

\[
z_n = \frac{q_n}{b_{n+2} b_{n+1}}, \quad n \in N.
\]
Therefore the general solution of (6) is
\[ x_n = x_1 q_1 \left[ \prod_{j=2}^{n-2} a_j \right] \sum_{(n-4,2)} z_i + x_2 \left[ \prod_{j=1}^{n-2} a_j \right] \sum_{(n-3,1)} z_i, \quad n \geq 2, \]
where \( z_n = q_n/a_n a_{n-1} \), \( n \geq 2 \), \( z_1 = q_1/a_1 \), and \( x_1, x_2 \) are arbitrary constants.

Having solved (6) we can obtain the general solution of a linear homogeneous second order equation
\[ a_n x_{n+2} + b_n x_{n+1} + c_n x_n = 0, \quad n \in \mathbb{N}, \]
where \( a, b : \mathbb{N} \rightarrow \mathbb{R} \setminus \{0\} \), and \( c : \mathbb{N} \rightarrow \mathbb{R} \). This solution is
\[ x_n = (-1)^n x_1 c_1 \left[ \prod_{j=2}^{n-2} b_j \right] \sum_{(n-4,2)} z_i \]
\[ + (-1)^n x_2 \left[ \prod_{j=1}^{n-2} b_j \right] \sum_{(n-3,1)} z_i, \quad n \geq 2, \]
where \( x_1, x_2 \) are arbitrary constants and
\[ z_n = -\frac{c_n a_{n-1}}{b_n b_{n-1}} \quad \text{for } n > 1. \]

Considering the linear nonhomogeneous equation we have the following result.

**THEOREM 2.** Let \( b, c : \mathbb{N} \rightarrow \mathbb{R} \setminus \{0\} \), \( r : \mathbb{N} \rightarrow \mathbb{R} \),
\[ \sum_{(n,1)} z_i \neq 0 \quad \text{for } n \in \mathbb{N}, \]
where
\[ z_n = \frac{-c_n}{b_n b_{n-1}}, \quad n \geq 2. \]

Then the general solution of
\[ x_{n+2} + b_n x_{n+1} + c_n x_n = r_n, \quad n \in \mathbb{N}, \]
is
\[ x_n = (-1)^n x_2 \left[ \prod_{j=1}^{n-2} b_j \right] \sum_{(n-3,1)} z_i \]
\[ + (-1)^n x_1 c_1 \left[ \prod_{j=1}^{n-2} b_j \right] \left[ \sum_{(n-3,1)} z_i \right] \left[ \prod_{j=2}^{n-1} \frac{1}{c_j} \right] \frac{1}{y_{k+1} y_{k+2}} \]
\[ + (-1)^n \left[ \prod_{j=1}^{n-2} b_j \right] \left[ \sum_{(n-3,1)} z_i \right] \left[ \prod_{j=2}^{n-1} \frac{1}{c_j} \right] \frac{1}{y_{k+1} y_{k+2}} \sum_{j=1}^{k} r_j y_j + 1 \]
for \( n \geq 2 \), where \( x_1, x_2 \) are arbitrary constants, and
\[ y_n = (-1)^n \left[ \prod_{j=1}^{n-2} \frac{b_j}{c_j+1} \right] \sum_{(n-3,1)} z_i, \quad n \geq 2. \]
PROOF. In order to obtain (12), we consider the equation

\begin{equation}
\label{eq:13}
c_{n+1}y_{n+2} + b_ny_{n+1} + y_n = 0, \quad n \in \mathbb{N}.
\end{equation}

The general solution of (13) is, by (8),

\begin{equation}
\label{eq:14}
y_n = (-1)^n \sum_{j=1}^{n-2} \frac{b_j}{c_{j+1}} \left[ \frac{1}{b_1} \sum_{i=1}^{n-4,2} z_i + y_2 \sum_{i=3,1}^{n-3,1} z_i \right],
\end{equation}

where \( z_n \) is defined by (10). Let \( y_n \) satisfy the initial conditions \( y_1 = 0, y_2 = 1 \). As a result, we get from (14)

\begin{equation}
\label{eq:15}
y_n = (-1)^n \sum_{j=1}^{n-2} \frac{b_j}{c_{j+1}} \sum_{i=3,1}^{n-3,1} z_i, \quad n \geq 2.
\end{equation}

Under the hypothesis (9), the solution (15) vanishes at no point \( n, n \geq 2 \).

On multiplying (11) by \( y_{n+1} \) and writing (13) in the form \( b_ny_{n+1} = -y_n - c_{n+1}y_{n+2} \) we have \( (y_{n+1}x_{n+2} - c_{n+1}y_{n+2}x_{n+1}) - (y_nx_{n+1} - c_ny_{n+1}x_n) = r_ny_{n+1} \). On summing the above equation with respect to \( n \) one gets

\begin{equation}
\label{eq:16}
y_{n+1}x_{n+2} - c_{n+1}y_{n+2}x_{n+1} = y_1x_2 - c_1y_2x_1 + \sum_{j=1}^{n} r_jy_{j+1}, \quad n \in \mathbb{N}.
\end{equation}

Setting \( y_1 = 0 \) and \( y_2 = 1 \), dividing (16) by \( y_{n+1}y_{n+2} \prod_{j=1}^{n} c_{j+1} \), and summing next with respect to \( n \), we have

\[
\frac{x_{n+2}}{y_{n+2}} \prod_{j=1}^{n} \frac{1}{c_{j+1}} - x_2 = -c_1x_1 \sum_{k=1}^{n} \prod_{j=1}^{k} \frac{1}{c_{j+1}} \frac{1}{y_{k+1}y_{k+2}} + \sum_{k=1}^{n} \prod_{j=1}^{k} \frac{1}{c_{j+1}} \frac{1}{y_{k+1}y_{k+2}} \sum_{j=1}^{k} r_jy_{j+1}.
\]

Thus

\[
x_{n+2} = x_2y_{n+2} \prod_{j=1}^{n} c_{j+1} - c_1x_1y_{n+2} \prod_{j=1}^{n} c_{j+1} \sum_{k=1}^{n} \prod_{j=1}^{k} \frac{1}{c_{j+1}} \frac{1}{y_{k+1}y_{k+2}} + y_{n+2} \prod_{j=1}^{n} c_{j+1} \sum_{k=1}^{n} \prod_{j=1}^{k+1} \frac{1}{c_{j+1}} \frac{1}{y_{k+1}y_{k+2}} \sum_{j=1}^{k} r_jy_{j+1}, \quad n \in \mathbb{N}.
\]

Hence, by (15) we infer that (12) holds for \( n > 2 \). Furthermore, by (iii) and (iv), (12) is also valid for \( n = 2 \). Q.E.D.

Note that equation (11) can possibly be solved even in the case when (9) is not fulfilled, that is, even if the \( y_n \) given by (15) vanished for some \( n \). The \( y_n \) given by (15) is one particular solution of (13). The general solution is given by (14). If the \( y_n \) given by (15) vanished for some \( n \geq 2 \), then any other particular solution
obtained from (14) which did not become zero for any \( n \in N \) could be used. This solution rather than (15) should be substituted in the following equality:

\[
x_{n+2} = y_{n+2} \left[ \prod_{j=2}^{n+1} c_j \right] \left\{ \frac{x_2}{y_2} + \sum_{k=1}^{n} \left[ \frac{1}{y_{k+1}y_{k+2}} \left( \prod_{j=2}^{k+1} \frac{1}{c_j} \right) \cdot \left( y_1 x_2 - c_1 y_2 x_1 + \sum_{j=1}^{k} r_j y_{j+1} \right) \right] \right\}, \quad n \in N.
\]

However, the obtained expression for the solution of (11) will be more complicated in any case.

Finally, we see that the equation

\[
(17) \quad a_n x_{n+2} + b_n x_{n+1} + c_n x_n = r_n,
\]

where \( a, b, c : N \to R \setminus \{0\} \), \( r : N \to R \), can be solved by applying Theorem 2.

On dividing (17) by \( a_n \), we get

\[
x_{n+2} + \frac{b_n}{a_n} x_{n+1} + \frac{c_n}{a_n} x_n = \frac{r_n}{a_n}
\]

which is of the type considered in Theorem 2. Therefore, to get its solution we need a nonvanishing solution of the equation

\[
\frac{c_{n+1}}{a_{n+1}} y_{n+2} + \frac{b_n}{a_n} y_{n+1} + y_n = 0.
\]

If for instance \( \sum_{n=1}^{n} z_i \neq 0 \) for all \( n \in N \), where \( z_n = -c_n a_{n-1}/b_n b_{n-1} \), we can write the general solution of (17) in the form

\[
x_n = (-1)^n \left[ \prod_{j=1}^{n-2} \frac{b_j}{a_j} \right] \left[ \sum_{(n-3,1)} z_i \right] \cdot \left\{ x_2 - \frac{c_1}{a_1} x_1 + \sum_{k=1}^{n-2} \left[ \prod_{j=2}^{k+1} \frac{a_j}{c_j} \right] \frac{1}{y_{k+1}y_{k+2}} \right. \\
+ \sum_{k=1}^{n-2} \left[ \prod_{j=2}^{k+1} \frac{a_j}{c_j} \right] \frac{1}{y_{k+1}y_{k+2}} \sum_{j=1}^{k} \frac{r_j}{c_j} y_{j+1} \right\}, \quad n > 2,
\]

where \( x_1, x_2 \) are arbitrary constants, \( z_n \) is defined above, and

\[
y_n = (-1)^n \frac{a_{n-1}}{a_1} \left[ \prod_{j=1}^{n-2} \frac{b_j}{c_{j+1}} \right] \sum_{(n-3,1)} z_i, \quad n \geq 2, \ y_1 = 0.
\]

3. Nonlinear examples. Various equations can be made linear by a change of variable. In particular, such a type is Riccati's equation

\[
(18) \quad a_n x_{n+1} x_n + b_n x_n + c_n = 0,
\]

where \( a, b, c : N \to R \setminus \{0\} \). Make the change of variable \( x_n = y_{n+1}/y_n \), with \( y_n \neq 0 \) for all \( n \in N \). Then (18) becomes

\[
a_n y_{n+2} + b_n y_{n+1} + c_n y_n = 0.
\]
Hence, by (8)

\[ x_n = -\frac{b_{n-1}}{a_{n-1}} \left\{ \frac{c_1 \sum_{(n-3,2)} z_i + b_1 x_1 \sum_{(n-2,1)} z_i}{c_1 \sum_{(n-4,2)} z_i + b_1 x_1 \sum_{(n-3,1)} z_i} \right\}, \quad n \geq 2, \]

where \( z_n = -c_n a_{n-1}/b_n b_{n-1}, \ n > 1, \ x_1 \) is an arbitrary constant.

For \( y_n \neq 0 \) we need

\[ c_1 \sum_{(n-3,2)} z_i + b_1 x_1 \sum_{(n-2,1)} z_i \neq 0 \quad \text{for all} \ n \in N. \]

The second example treats the equation

\[ a_n x_{n+1} x_n + b_n x_{n+1} + c_n = 0. \tag{19} \]

Here, by putting \( x_n = y_n/y_{n+1}, \ y_n \neq 0 \) on \( N \), we get

\[ c_n y_{n+2} + b_n y_{n+1} + a_n y_n = 0. \]

Supposing \( a, b, c : N \to \mathbb{R} \setminus \{0\} \), and

\[ a_1 x_1 \sum_{(n-4,2)} z_i + b_1 \sum_{(n-3,1)} z_i \neq 0 \quad \text{for all} \ n \in N, \]

where \( z_n = -a_n c_{n-1}/b_n b_{n-1}, \ n > 1, \) we obtain, by (8), the expression of the solution of the considered equation. That is, the solution \( x_n \) of (19) is given by

\[ x_n = -\frac{c_{n-1}}{b_{n-1}} \left\{ \frac{a_1 x_1 \sum_{(n-4,2)} z_i + b_1 \sum_{(n-3,1)} z_i}{a_1 x_1 \sum_{(n-3,2)} z_i + b_1 \sum_{(n-2,1)} z_i} \right\}, \quad n \geq 2. \]

4. Note that the operators introduced in this paper can be used with some modification for recurrence equations of higher order.

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