PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 100, Number 1, May 1987

ON THE ALMOST EVERYWHERE CONVERGENCE TO L^p DATA FOR HIGHER ORDER HYPERBOLIC OPERATORS

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ABSTRACT. First we prove a sharp maximal Fourier integral theorem for $L^p(\mathbf{R}^n)$, $1 < p \leq 2$, using the techniques of [**4–6**]. Then we apply the maximal theorem to prove a sharp result concerning the almost everywhere convergence to L^p-initial data for the Cauchy problem for smooth variable coefficient strictly hyperbolic linear partial differential operators of order $m > 2$.

1. Introduction. The purpose of this note is to show how a recent interpolation idea of Rubio de Francia [**4**] can be used along with results in [**6**, **7**] to extend Stein's [**9**] theorem regarding the Cauchy problem for second order strictly hyperbolic operators to such operators of higher order.

We will be dealing with operators of the form

$$L = \partial^m/\partial t^m - \sum_{j=1}^{m} A_{m-j}(t, x, \partial_x)\partial^j/\partial t^j,$$

where A_{m-j} is a differential operator of order $m-j$ with smooth coefficients which, for simplicity, we assume to be constant off of a compact set. One could relax this condition by assuming instead a finite domain of dependence condition as in [**6**]. We recall that such an operator is said to be strictly hyperbolic if the principal symbol, $L_m(\xi, \tau)$, has the property that when $\xi \in \mathbf{R}^n\backslash 0$, the polynomial in τ, $L_m(\xi, \tau)$, has distinct real roots. For such operators we will study the following Cauchy problem for $\mathbf{R}^{n+1}_+$:

$$(1.1) \qquad \begin{cases} Lu = 0, & t > 0, \\ (\partial^j/\partial t^j)u(x,0) = 0, & 1 \leq j < m-1, \\ (\partial^{m-1}/\partial t^{m-1})u(x,0) = f. & \end{cases}$$

Our chief result will be that if $m \geq 2$ and if L is as above, then for $f \in L^p_{\text{loc}}(\mathbf{R}^n)$, $p > p_{n,m} = \max\{(1, 2n/(n+2m-3)\}$, the (weak) solution to (1.1) has the property that

$$(1.2) \qquad u(x,t)/t^{m-1} \to f(x)/(m-1)!$$

almost everywhere. Also we will show that this result is sharp in the sense that if $p_{n,m} \geq 1$, there are operators L for which there is not almost everywhere convergence to $L^{p_{n,m}}(\mathbf{R}^n)$ initial data.

When $m = 2$ this result in the constant coefficient case is due to Stein [**9**] and in the variable coefficient case to Greenleaf [**3**] and Ruiz [**5**]. Also when $m \geq 3$ it was shown by the author [**6**] that there is almost everywhere convergence to L^2_{loc} data.

Received by the editors March 12, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35L15, 42B25.
The author was supported in part by an NSF postdoctoral fellowship.

To prove our result we will first establish a maximal Fourier integral theorem for $L^p(\mathbf{R}^n)$, $1 < p \leq 2$, and then apply it along with the Fourier integral representation for solutions to (1.1). Since the proof of the L^p maximal Fourier integral theorem is very similar to the proof of the L^2 maximal Fourier integral theorem in [**6**] we will only sketch the details of the proof.

2. An L^p maximal Fourier integral theorem. Suppose that $a(t; x, \xi)$ and $\phi(t; x, \xi)$ are smooth on $\mathbf{R}_+ \times \mathbf{R}^n \backslash 0$ and set

$$(T_t f)(x) = \int_{\mathbf{R}^n} \hat{f}(\xi) a(t; x, t\xi) \exp\left[i(\langle x, \xi\rangle + t\phi(t; x, \xi))\right] d\xi. \tag{2.1}$$

Assume that $a(t; x, \xi) = 0$ when $|\xi| \leq 2$ and also that for some $\delta > \frac{1}{2}$ one has

$$|(\partial/\partial t)^j (D_x^\alpha)(D_\xi^\beta) a(t; x, \xi)| \leq C|\xi|^{-\delta - |\beta|}, \qquad j \leq 1,\ |\alpha|, |\beta| \leq n+3. \tag{2.2}$$

Also the phase functions ϕ are real, homogeneous of degree 1 in ξ, and satisfy

$$|(\partial/\partial t)^j (D_x^\alpha)(D_\xi^\beta) \phi(t; x, \xi)| \leq 1/2 \quad \text{if } |\xi| = 1,\ j \leq 1,\ |\alpha|, |\beta| \leq n+3. \tag{2.3}$$

Then our result is the following

THEOREM 2.1. *Let $n \geq 1$. If $1 < p \leq 2$ and if $\delta > \delta(p) = [2n - p(n-1)]/2$ in (2.2) then*

$$\left\| \sup_{0 < t \leq 1} |(T_t f)(x)| \right\|_{L^p(\mathbf{R}^n)} \leq C \|f\|_{L^p(\mathbf{R}^n)}. \tag{2.4}$$

We remark that when $p = 2$ this result in the multiplier case is due independently to Bourgain [**1**] and Sogge-Stein [**7**], and also for $1 < p < 2$ the multiplier case is due to Rubio de Francia [**4**]. When $p = 2$ the inequality (2.4) was proved by the author in [**6**].

To prove (2.4) we will, as in Rubio de Francia [**4**], break up the amplitude $a(t; x, \xi)$ dyadically and use square function arguments to obtain estimates for each of the resulting pieces. For this purpose let $\psi \in C_0^\infty(\mathbf{R})$ satisfy $\operatorname{supp}(\psi) \subset [\frac{1}{2}, \frac{3}{2}]$ and $\sum_{j=0}^\infty \psi(2^{-j}s) = 1$ for $s \geq 2$, and set $a_j(t; x, \xi) = \psi(2^{-j}|\xi|) a(t; x, \xi)$. Then if we define $T_{j,t}$ to be the operator where a is replaced by a_j in (2.1) and if $(T_j^* f)(x) = \sup_{0<t\leq 1} |(T_{j,t} f)|$, then to prove (2.4) it is enough to show that if $\delta > \frac{1}{2}$ in (2.4) is arbitrary then for every $\varepsilon > 0$ one has

$$\left\| T_j^* f \right\|_p \leq C_{p,\varepsilon} 2^{-j(\delta - \delta(p) - \varepsilon)} \|f\|_p. \tag{2.5}$$

However,

$$\begin{aligned} (T_j^* f)^2 &\leq 2 \int_0^1 |T_{j,t} f| \left| \frac{d}{dt} T_{j,t} f \right| dt \\ &= 2 \int_0^1 |T_{j,t} f| \left| \tilde{T}_{j,t} f \right| dt/t \leq 2 S_j f \cdot \tilde{S}_j f. \end{aligned} \tag{2.6}$$

Here $\tilde{T}_j$ is a Fourier integral operator whose amplitude

$$\tilde{a}_j(t; x, \xi) = \psi(2^{-j}|\xi|) \tilde{a}(t; x, \xi)$$

only satisfies the estimates (2.2) with $\tilde{\delta} = \delta + 1$, where δ is the exponent for a. Also $S_j f$ is the square function

$$S_j f = \left(\int_0^1 |T_{j,t} f|^2 \, dt/t \right)^{1/2},$$

and $\tilde{S}_j f$ is similarly defined.

Now if one uses the standard L^2 theory of Fourier integrals as in [**6**] it is easy to verify that one has the uniform estimates

$$\|S_j f\|_2 \le C 2^{-j\delta} \|f\|_2, \quad \|\tilde{S}_j f\|_2 \le C 2^{-j(\delta-1)} \|f\|_2, \tag{2.7}$$

and hence by (2.6) one has

LEMMA 2.1. *For every $j \le 1$,*

$$\|T_j^* f\|_2 \le C 2^{j(1/2-\delta)} \|f\|_2.$$

As a result of this it follows from real interpolation that (2.5) is a consequence of the following estimate.

LEMMA 2.2. *If $\beta > n/2$, then for every $j \ge 1$*

$$\|T_j^* f\|_{L^1} \le C_\beta 2^{-j(\delta-\beta-1/2)} \|f\|_{H^1}.$$

However from (2.6) it follows that $T_j^* f \le 2^{j/2} S_j f + 2^{-j/2} \tilde{S}_j f$, so one now only needs to verify that if $\beta > n/2$, then

$$\|S_j f\|_{L^1} \le C 2^{j(\beta-\delta)} \|f\|_{H^1}, \qquad \|\tilde{S}_j f\|_{L^1} \le C 2^{j(1+\beta-\delta)} \|f\|_{H^1}. \tag{2.8}$$

These two estimates, however, follow easily from the proof of the Hörmander multiplier theorem (cf. [**11**, pp. 271–275; **4**]). In fact, if

$$K_t(x,y) = \int \psi(2^{-j} |t\xi|) \, a(t; x, t\xi) \, e^{i[\langle y,\xi\rangle + t\phi(t;x,\xi)]} \, d\xi,$$

then by Calderón-Zygmund theory (since (2.7) holds) one only needs to verify that there is a constant C_β depending only on $\beta > n/2$ such that

$$\int_{|x|>2|y|} \left(\int_0^1 |K_t(x + x_0, x - y) - K_t(x + x_0, x)|^2 \, dt/t \right)^{1/2} dx \le C_\beta 2^{-j(\delta-\beta)},$$

as well as a similar estimate for the kernels $\tilde{K}_t$ of $\tilde{T}_{j,t}$. These two estimates follow in a straightforward manner using the L^2-boundedness of Fourier integral operators and so the details are left to the reader.

3. Almost everywhere convergence in the Cauchy problem. We are now in a position to use the Fourier integral representation for solutions to (1.1) in order to obtain our main result. Recall that L is a strictly hyperbolic differential operator of order $m \ge 2$ on $\mathbf{R}^{n+1}$ whose smooth coefficients are constant outside of a compact set.

THEOREM 3.1. *Let L be an mth order strictly hyperbolic operator as above with $m \geq 2$. Then if $f \in L^p_{\text{loc}}(\mathbf{R}^n)$, $p > p_{n,m} = \max\{1, 2n/(n+2m-3)\}$, the (weak) solution to the Cauchy problem (1.1) has the property that*

$$u(x,t)/t^{m-1} \to f(x)/(m-1)! \tag{3.1}$$

almost everywhere. Furthermore, this result is sharp in the sense that if $p_{n,m} \geq 1$, there are operators L as above for which (3.1) does not hold on $L^{p_{n,m}}(\mathbf{R}^n)$.

As one can easily verify, (3.1) follows from Theorem 2.1 and the following well-known result for small time solutions of (1.1) (see eg. [**12**, pp. 308–313; **2**, pp. 215–221]).

LEMMA 3.1. *Let L be as above. Then there is a $t_0 > 0$ so that for $0 < t \leq t_0$ the solution $u(x,t)$ of the Cauchy problem (1.1) satisfies*

$$\frac{u(x,t)}{t^{m-1}} = \sum_{k=0}^{m-1} \int \hat{f}(\xi)\, a_k(t;x,t\xi)\, e^{-2\pi i[\langle x,\xi\rangle + t\phi_k(t;x,\xi)]}\, d\xi + Rf(t,x).$$

Here $\phi_k(t;x,\xi)$ and $a_k(t;x,\xi)$ are smooth on $(0,t_0] \times \mathbf{R}^n \times \mathbf{R}^n\backslash 0$, $a_k(t;x,\xi) = 0$ if $|\xi| \leq 2$, and a_k satisfies (2.2) with $\delta = m-1$. Also $\phi_k(t;x,\xi)$ is real, homogeneous of degree one in ξ, and satisfies

$$|(\partial/\partial t)^j (D_x^\alpha)(D_\xi^\beta)\phi_k(t;x,\xi)| \leq C$$

if $|\xi| = 1$, $j \leq 1$, and $|\alpha|, |\beta| \leq n+3$. Finally, $|(Rf)(t,x)| \leq Cf^*(x)$, where f^* is the Hardy-Littlewood maximal function for f.

Since, as we have noted, Lemma 3.1 implies (3.1) we are only left with demonstrating the sharpness of Theorem 3.1. To this end let $l \geq 1$ and $n \geq 2$ and consider the strictly hyperbolic operator L of order $m = 2l$,

$$L = \prod_{j=1}^{l} \left(\frac{\partial^2}{\partial t^2} - j\Delta_x \right).$$

Then the (weak) solution to the Cauchy problem (1.1) must have the form $u(x,t)/t^{m-1} = \sum_{j=1}^{l} K_{j,t} * f + R_t f$, where $K_{j,t}(x) = K_j(x/t) \cdot t^{-n}$ with

$$K_j = c_j \left[\chi(\xi) \sin(2\pi j\,|\xi|) \,/\, |\xi|^{m-1} \right]^{\vee}$$

and $|R_t f| \leq Cf^*$ (see [**2**, pp. 215–221]). Thus, if $p_{n,m} = 2n/(n+2m-3) \geq 1$, it follows from counterexamples of Stein-Wainger [**10**] that there is an $f \in L^{p_{n,m}}(\mathbf{R}^n)$ for which $\sup_{t>0} |\sum_{j=1}^{l} (K_{j,t} * f)(x)| = \infty$ for all x. In a similar manner one can show that when $m = 2l+1$ is odd, the results for

$$L = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x_1} \right) \cdot \prod_{j=1}^{l} \left(\frac{\partial^2}{\partial t^2} - (j+1)\Delta_x \right)$$

must be sharp. Also, if $n = 1$, then $p_{1,m} \geq 1$ only when $m = 2$, and one can easily check that the above results are sharp when $L = \partial^2/\partial t^2 - \partial^2/\partial x^2$.

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