ON THE STABILITY OF EXPONENTIAL BASES IN $L^2(-\pi, \pi)$

ROBERT M. YOUNG

ABSTRACT. In this note we investigate the effects of small perturbations on an important class of exponential bases.

1. Introduction. A system of complex exponentials $\{e^{i\lambda_n t}\}$ is said to be "equivalent" to the trigonometric system $\{e^{int}\}$ in $L^2(-\pi, \pi)$—or a Riesz basis—if the mapping $e^{int} \to e^{i\lambda_n t}$ ($-\infty < n < \infty$) can be extended to an isomorphism on all of $L^2(-\pi, \pi)$. If this is the case, then each function $f$ in $L^2(-\pi, \pi)$ will have a unique nonharmonic Fourier expansion

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{i\lambda_n t} \quad \text{(in the mean)}$$

with $\{c_n\}$ in $l^2$. Furthermore, there are positive constants $A$ and $B$ depending only on $\{\lambda_n\}$, and not on $f$, such that

$$A \sum |c_n|^2 \leq ||f||^2 \leq B \sum |c_n|^2. \quad (1)$$

The study of nonharmonic Fourier series was initiated by Paley and Wiener who showed that the system $\{e^{i\lambda_n t}\}$ is a Riesz basis for $L^2(-\pi, \pi)$ whenever each $\lambda_n$ is real and $|\lambda_n - n| \leq L < 1/\pi^2$. Ultimately, Kadec showed that the constant $1/\pi^2$ could be replaced by $1/4$. That $1/4$ is in fact the "best possible" constant follows from the fact that the system $\{e^{\pm it(n-1/4)t} : n = 1, 2, 3, \ldots\}$ is already complete in $L^2(-\pi, \pi)$. (For a comprehensive introduction to the theory of nonharmonic Fourier series, including proofs of these assertions, see [10].)

It is well known that any Riesz basis of complex exponentials—and not just the trigonometric system—remains stable under sufficiently small perturbations of its exponents. In fact, much more is true.

THEOREM 1. If the system $\{e^{i\lambda_n t}\}$ is an unconditional basis for $L^2(-\pi, \pi)$, then there is a positive constant $L$ such that $\{e^{i\mu_n t}\}$ is also an unconditional basis for $L^2(-\pi, \pi)$ whenever $|\lambda_n - \mu_n| \leq L$ for every $n$.

Recall that a sequence $\{f_n\}$ in a separable Hilbert space $H$ is said to be an unconditional basis for $H$ if each element $f$ in the space has a unique expansion

$$f = \sum c_n f_n$$

in which the series is "unconditionally convergent," that is, it converges to the same sum after any permutation of its terms. It is clear from (1) that every Riesz basis

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is both bounded and unconditional. Remarkably, the converse is also true: In a separable Hilbert space all bounded unconditional bases are equivalent. (For a quick, elegant proof see [7, pp. 18, 71].) Therefore, if \( \{f_n\} \) is an unconditional basis for \( H \) then \( \{f_n/\|f_n\|\} \) is both unconditional and bounded, and we have, corresponding to (1), the “approximate Parseval identity”

\[
A \sum |c_n|^2 \|f_n\|^2 \leq \sum c_n f_n \|f_n\|^2 \leq B \sum |c_n|^2 \|f_n\|^2.
\]

Unconditional bases constitute the largest and most tractable class of bases known. In fact it is surprisingly difficult to exhibit a basis for \( L^2(-\pi, \pi) \) that is not unconditional. (The first example of such a basis was given in 1948 by Babenko. For an account of this and other “conditional” bases in Hilbert space see [9].)

At present it is not known whether there exists a conditional basis of complex exponentials.

Theorem 1 was established in [3, p. 295] but the proof is extremely difficult, requiring deep properties of Hardy spaces of analytic functions and Toeplitz operators. Even in the special case in which the basis \( \{e^{i\lambda_n t}\} \) is both bounded and unconditional—that is to say, a Riesz basis—the proof relies on nontrivial facts about “nonharmonic Fourier frames” [10, p. 191]. Nevertheless, we show in the next section that Theorem 1 is in fact completely elementary and we offer a new and straightforward proof.

As was pointed out, Kadec’s “\( \frac{A}{2} \)-Theorem” does not remain valid in the limiting case \( |\lambda_n - n| \leq \frac{1}{4} \), for the trivial reason that these inequalities permit the system \( \{e^{i\lambda_n t}\} \) to be overcomplete in \( L^2(-\pi, \pi) \). An important and widely studied example in which the corresponding exponentials are not overcomplete is given by the sequence

\[
\lambda_n = \begin{cases} 
  n + \frac{1}{4}, & n > 0, \\
  0, & n = 0, \\
  n - \frac{1}{4}, & n < 0.
\end{cases}
\]

It is well known that for this choice of exponents the system \( \{e^{i\lambda_n t}\} \) is exact in \( L^2(-\pi, \pi) \) and yet not a Riesz basis (see, e.g., [4 and 8]). On the other hand, it was shown in [11] that each element \( f \in L^2(-\pi, \pi) \) has a unique nonharmonic expansion

\[
f \sim \sum c_n e^{i\lambda_n t}
\]

which is equiconvergent with its ordinary Fourier series, uniformly on every closed subinterval of \( (-\pi, \pi) \). (Recall that two series \( \sum a_n \) and \( \sum b_n \) are said to be equiconvergent if their difference \( \sum (a_n - b_n) \) converges and has the sum zero.) Since the Fourier series of an \( L^2 \) function converges to the function pointwise almost everywhere, the same must be true for the nonharmonic series. Thus there was some hope that the system \( \{e^{i\lambda_n t}\} \), with the \( \lambda_n \) given by (2), might prove to be a basis for \( L^2(-\pi, \pi) \). Using an argument developed in [12], we show that, unfortunately, this is not the case.

**Theorem 2.** The system \( \{e^{i\lambda_n t}\} \), where the \( \lambda_n \) are given by (2), is not a basis for \( L^2(-\pi, \pi) \).

The search for a conditional basis of complex exponentials continues.
2. Proof of Theorem 1. Suppose that the system \( \{e^{i\lambda_nt}\} \) is an unconditional basis for \( L^2(-\pi, \pi) \) and that the inequalities

\[
A \sum |c_n|^2\|e^{i\lambda_nt}\|^2 \leq \left\| \sum c_n e^{i\lambda_nt} \right\|^2 \leq B \sum |c_n|^2\|e^{i\lambda_nt}\|^2
\]

hold for every convergent expansion \( \sum c_n e^{i\lambda_nt} \). Let \( \{\mu_n\} \) satisfy \( |\lambda_n - \mu_n| \leq L \). It is to be shown that if \( L \) is sufficiently small, then

\[
\left\| \sum c_n (e^{i\lambda_nt} - e^{i\mu_nt}) \right\| \leq \theta \left\| \sum c_n e^{i\lambda_nt} \right\|
\]

where \( 0 \leq \theta < 1 \) and \( \{c_n\} \) is any finite sequence of scalars. This simple criterion ensures that the mapping \( e^{i\lambda_nt} \rightarrow e^{i\mu_nt} \) can be extended to a bounded linear operator \( T \) on all of \( L^2(-\pi, \pi) \) and that \( \|I - T\| \leq \theta < 1 \) (see, e.g., [9, p. 84]). Thus \( T \) is an isomorphism and \( \{e^{i\mu_nt}\} \) is a basis for \( L^2(-\pi, \pi) \) equivalent to \( \{e^{i\lambda_nt}\} \).

To establish (4) we first write

\[
e^{i\mu_nt} - e^{i\lambda_nt} = e^{i\lambda_nt}(e^{i(\mu_n - \lambda_n)t} - 1)
\]

and then expand \( e^{i\lambda_nt} \) in an everywhere-convergent Taylor series. Inasmuch as \( \|t^kg(t)\| \leq \pi^k\|g(t)\| \), we have for every finite sequence \( \{c_n\} \)

\[
\left\| \sum c_n (e^{i\lambda_nt} - e^{i\mu_nt}) \right\| = \left\| \sum c_n \sum_{k=1}^{\infty} \frac{i(\mu_n - \lambda_n)^k}{k!} t^k e^{i\lambda_nt} \right\|
\leq \sum_{k=1}^{\infty} \frac{\pi^k}{k!} \left\| \sum c_n [i(\mu_n - \lambda_n)]^k e^{i\lambda_nt} \right\|
\leq \sum_{k=1}^{\infty} \frac{\pi^k}{k!} \left\{ B \sum_n |c_n|^2 |\lambda_n - \mu_n|^{2k} \|e^{i\lambda_nt}\|^2 \right\}^{1/2}
\leq \sum_{k=1}^{\infty} \frac{\pi^k}{k!} L^k \left\{ B \sum_n |c_n|^2 \|e^{i\lambda_nt}\|^2 \right\}^{1/2}
\leq \sum_{k=1}^{\infty} \frac{(\pi L)^k}{A} \frac{B}{A} \left\| \sum c_n e^{i\lambda_nt} \right\|
= \frac{B}{A} \left( e^{\pi L} - 1 \right) \left\| \sum c_n e^{i\lambda_nt} \right\|.
\]

Notice that we have made use of both inequalities in (3). Setting \( \theta = (B/A)(e^{\pi L} - 1) \), we have \( \theta < 1 \) provided \( L \) is sufficiently small.

Remark. The argument just given is similar to the one used by Duffin and Eachus [2] to show that \( \{e^{i\mu_nt}\} \) is a Riesz basis for \( L^2(-\pi, \pi) \) whenever \( |\mu_n - n| \leq L < (\log 2)/\pi \). In this case \( \lambda_n = n \) and Parseval’s identity shows that \( A = B = 1 \); therefore (4) holds with \( \theta = e^{\pi L} - 1 < 1 \). By expanding the function \( e^{i\lambda_nt} \), not as a Taylor series, but in terms of the orthonormal basis \( \{1, \cos nt, \sin(n - \frac{1}{2})t\}_{n=1}^{\infty} \), Kadec [5] deduced in the same way that (4) holds with \( \theta = 1 - \cos \pi L + \sin \pi L \), provided every \( \mu_n \) is real. Clearly \( L < \frac{1}{4} \) implies \( \theta < 1 \).
3. **Proof of Theorem 2.** We argue by contradiction. If the system were a basis, then we could write

\[(5) \quad \sin t = \sum c_n e^{i\lambda_n t} \quad \text{(in the mean)}.\]

To compute the \(c_n\), we shall make use of the Paley-Wiener space \(P\) consisting of all entire functions of exponential type at most \(\pi\) that are square integrable on the real axis. The inner product of two functions \(F\) and \(G\) in \(P\) is, by definition,

\[(F, G) = \int_{-\infty}^{\infty} F(x)G(x)\, dx.\]

By virtue of the Paley-Wiener theorem, the complex Fourier transform

\[f(t) \rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-i\omega t}\, dt\]

is an isometric isomorphism from \(L^2(-\pi, \pi)\) onto all of \(P\). The exponentials \(e^{i\lambda_n t}\) are sent to the "reproducing functions"

\[K_n(z) = \frac{\sin \pi(z - \lambda_n)}{\pi(z - \lambda_n)},\]

which then constitute a basis for \(P\). Let \(\{g_n\}\) be biorthogonal to \(\{K_n\}\) in \(P\). If we write \(\sin t = (e^{it} - e^{-it})/2i\) and then apply the Fourier transform to both sides of (5), the result is

\[\frac{1}{2i} \left[ \frac{\sin \pi(z - 1)}{\pi(z - 1)} - \frac{\sin \pi(z + 1)}{\pi(z + 1)} \right] = \sum c_n K_n(z).\]

Taking the inner product of each side with \(g_n\), and remembering that

\[\frac{\sin \pi(z - w)}{\pi(z - w)}/\pi(z - w)\]

is the reproducing kernel for \(P\), we find

\[c_n = \frac{1}{2i} [g_n(1) - g_n(-1)].\]

Now, the \(g_n\) can be determined explicitly. Let \(F(z)\) be the canonical product with zeros \(\lambda_n:\)

\[F(z) = z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{\lambda_n^2} \right).\]

It was shown by Levinson [6, p. 11] that

\[F(z) = \int_{-\pi}^{\pi} f(t)e^{izt}\, dt,\]

where \(f(t)\) is a multiple of \((\cos \frac{1}{2}t)^{-1/2}\sin \frac{1}{2}t\). Since \(f(t)\) is integrable over \((-\pi, \pi)\), it follows that \(F(z)\) is bounded along the real axis, and therefore each of the functions

\[F'(\lambda_n)(z - \lambda_n)\]
belongs to $P$. Since $(F_n, K_m) = F_n(\lambda_m) = \delta_{mn}$, it follows that the system $\{F_n\}$ is also biorthogonal to $\{K_n\}$ in $P$. But $\{K_n\}$ is complete and so has a unique biorthogonal sequence. Therefore, $F_n = g_n$ for every $n$, and we find
\[ c_n = \frac{1}{2i}[F_n(1) - F_n(-1)] = \frac{F(1)}{2iF'(\lambda_n)} \left[ \frac{1}{1 - \lambda_n} - \frac{1}{1 + \lambda_n} \right] \]
\[ = \frac{iF(1)}{F'(\lambda_n)} \left[ \frac{\lambda_n}{\lambda_n^2 - 1} \right]. \]

Notice that $c_0 = 0$ and, since $F'(z)$ is even, $c_{-n} = -c_n$ for every $n$. Accordingly, (5) becomes
\[ \sin t = -2F(1) \sum_{n=1}^{\infty} \frac{\sin \lambda_n t}{\lambda_n F'(\lambda_n)} \left[ \frac{\lambda_n^2}{\lambda_n^2 - 1} \right]. \]

Now the values of $F'(\lambda_n)$ were determined explicitly in [8]:
\[ F'(\lambda_n) = (-1)^n \lambda_n \Gamma \left( \frac{5}{4} \right) \frac{\Gamma(n)}{\Gamma(n + \frac{3}{2})} (n = 1, 2, \ldots). \]

Using the asymptotic formula $\Gamma(n)/\Gamma(n + \frac{3}{2}) = n^{-3/2} + O(n^{-5/2})$ [1], we have
\[ F'(\lambda_n) = A(-1)^n \lambda_n^{-1/2} + \varepsilon_n, \]
where $\varepsilon_n = O(1/n^{3/2})$ as $n \to \infty$. Since $|\lambda_n F'(\lambda_n)|$ tends to infinity with $n$, we see that the series in (6) differs from
\[ \sum_{n=1}^{\infty} \frac{\sin \lambda_n t}{\lambda_n F'(\lambda_n)} \]
by a series which converges uniformly on $[-\pi, \pi]$, and therefore our task is to show that (8) diverges in $L^2(-\pi, \pi)$. But if we apply the asymptotic formula (7) once again, we see that the difference between (8) and
\[ \frac{1}{A} \sum_{n=1}^{\infty} \frac{(-1)^n \sin \lambda_n t}{\sqrt{\lambda_n}} \]
is also uniformly convergent on $[-\pi, \pi]$. Accordingly, it suffices to show that (9) diverges in $L^2(-\pi, \pi)$.

Let $x = \pi - t$ ($0 \leq t \leq \pi$). Then
\[ \sum_{n=1}^{\infty} \frac{(-1)^n \sin \lambda_n t}{\sqrt{\lambda_n}} = \sum_{n=1}^{\sin(\pi/4 - \lambda_n x)} \frac{\sin(\pi/4 - \lambda_n x)}{\sqrt{\lambda_n}}. \]

For $N = 1, 2, 3, \ldots$, let $I_N$ be the interval $[0, \pi/32N]$. If $x \in I_N$ then $\pi/4 - \lambda_n x$ lies in the interval $[\pi/8, \pi/4]$ whenever $1 \leq n \leq 2N$, and hence, for these $n$, $\sin(\pi/4 - \lambda_n x) \geq B > 0$. Thus
\[ \sum_{N=1}^{2N} \frac{\sin(\pi/4 - \lambda_n x)}{\sqrt{\lambda_n}} \geq B \sum_{N=1}^{2N} \frac{1}{\sqrt{n + 1/4}} \geq B \frac{N + 1}{\sqrt{2N + 1/4}} \geq C\sqrt{N}, \]
where $C$ is a positive constant independent of $N$. Accordingly
\[
\left\| \sum_{N} \frac{(-1)^n \sin \lambda_n t}{\sqrt{\lambda_n}} \right\|^2 \geq \left\| \sum_{N} \frac{\sin \left( \frac{\pi}{4} - \lambda_n x \right)}{\sqrt{\lambda_n}} \right\|^2_{L^2(I_N)} \\
\geq \frac{1}{2\pi} C^2 N \left( \frac{\pi}{32N} \right) = \frac{C^2}{64}
\]
for all $N$. This shows that the series in (5) does not converge in $L^2(-\pi, \pi)$ and hence that the system $\{e^{i\lambda_n t}\}$ fails to be a basis.

REFERENCES