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ON THE STABILITY OF EXPONENTIAL BASES IN $L^2(-\pi,\pi)$

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ABSTRACT. In this note we investigate the effects of small perturbations on an important class of exponential bases.

1. Introduction. A system of complex exponentials $\{e^{i\lambda_n t}\}$ is said to be "equivalent" to the trigonometric system $\{e^{int}\}$ in $L^2(-\pi,\pi)$—or a *Riesz basis*—if the mapping $e^{int} \to e^{i\lambda_n t}$ $(-\infty < n < \infty)$ can be extended to an isomorphism on all of $L^2(-\pi,\pi)$. If this is the case, then each function f in $L^2(-\pi,\pi)$ will have a unique *nonharmonic* Fourier expansion

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{i\lambda_n t} \quad \text{(in the mean)}$$

with $\{c_n\}$ in l^2. Furthermore, there are positive constants A and B depending only on $\{\lambda_n\}$, and not on f, such that

$$A\sum |c_n|^2 \le \|f\|^2 \le B\sum |c_n|^2. \tag{1}$$

The study of nonharmonic Fourier series was initiated by Paley and Wiener who showed that the system $\{e^{i\lambda_n t}\}$ is a Riesz basis for $L^2(-\pi,\pi)$ whenever each λ_n is real and $|\lambda_n - n| \le L < 1/\pi^2$. Ultimately, Kadec showed that the constant $1/\pi^2$ could be replaced by $\frac{1}{4}$. That $\frac{1}{4}$ is in fact the "best possible" constant follows from the fact that the system $\{e^{\pm i(n-1/4)t}\colon n = 1,2,3,\dots\}$ is already complete in $L^2(-\pi,\pi)$. (For a comprehensive introduction to the theory of nonharmonic Fourier series, including proofs of these assertions, see [**10**].)

It is well known that any Riesz basis of complex exponentials—and not just the trigonometric system—remains stable under sufficiently small perturbations of its exponents. In fact, much more is true.

THEOREM 1. *If the system $\{e^{i\lambda_n t}\}$ is an unconditional basis for $L^2(-\pi,\pi)$, then there is a positive constant L such that $\{e^{i\mu_n t}\}$ is also an unconditional basis for $L^2(-\pi,\pi)$ whenever $|\lambda_n - \mu_n| \le L$ for every n.*

Recall that a sequence $\{f_n\}$ in a separable Hilbert space H is said to be an *unconditional basis* for H if each element f in the space has a unique expansion

$$f = \sum c_n f_n$$

in which the series is "unconditionally convergent," that is, it converges to the same sum after any permutation of its terms. It is clear from (1) that every Riesz basis

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is both bounded and unconditional. Remarkably, the converse is also true: *In a separable Hilbert space all bounded unconditional bases are equivalent.* (For a quick, elegant proof see [**7**, pp. 18, 71].) Therefore, if $\{f_n\}$ is an unconditional basis for H then $\{f_n/\|f_n\|\}$ is both unconditional and bounded, and we have, corresponding to (1), the "approximate Parseval identity"

$$A\sum |c_n|^2\|f_n\|^2 \leq \left\|\sum c_n f_n\right\|^2 \leq B\sum |c_n|^2\|f_n\|^2.$$

Unconditional bases constitute the largest and most tractable class of bases known. In fact it is surprisingly difficult to exhibit a basis for $L^2(-\pi,\pi)$ that is not unconditional. (The first example of such a basis was given in 1948 by Babenko. For an account of this and other "conditional" bases in Hilbert space see [**9**].)

At present it is not known whether there exists a conditional basis of complex exponentials.

Theorem 1 was established in [**3**, p. 295] but the proof is extremely difficult, requiring deep properties of Hardy spaces of analytic functions and Toeplitz operators. Even in the special case in which the basis $\{e^{i\lambda_n t}\}$ is both bounded and unconditional—that is to say, a Riesz basis—the proof relies on nontrivial facts about "nonharmonic Fourier frames" [**10**, p. 191]. Nevertheless, we show in the next section that Theorem 1 is in fact completely elementary and we offer a new and straightforward proof.

As was pointed out, Kadec's "$\frac{1}{4}$-Theorem" does not remain valid in the limiting case $|\lambda_n - n| \leq \frac{1}{4}$, for the trivial reason that these inequalities permit the system $\{e^{i\lambda_n t}\}$ to be *overcomplete* in $L^2(-\pi,\pi)$. An important and widely studied example in which the corresponding exponentials are not overcomplete is given by the sequence

$$\lambda_n = \begin{cases} n+\frac{1}{4}, & n>0, \\ 0, & n=0, \\ n-\frac{1}{4}, & n<0. \end{cases} \tag{2}$$

It is well known that for this choice of exponents the system $\{e^{i\lambda_n t}\}$ is exact in $L^2(-\pi,\pi)$ and yet not a Riesz basis (see, e.g., [**4** and **8**]). On the other hand, it was shown in [**11**] that each element $f \in L^2(-\pi,\pi)$ has a unique nonharmonic expansion

$$f \sim \sum c_n e^{i\lambda_n t}$$

which is *equiconvergent* with its ordinary Fourier series, uniformly on every closed subinterval of $(-\pi,\pi)$. (Recall that two series $\sum a_n$ and $\sum b_n$ are said to be equiconvergent if their difference $\sum(a_n - b_n)$ converges and has the sum zero.) Since the Fourier series of an L^2 function converges to the function pointwise almost everywhere, the same must be true for the nonharmonic series. Thus there was some hope that the system $\{e^{i\lambda_n t}\}$, with the λ_n given by (2), might prove to be a basis for $L^2(-\pi,\pi)$. Using an argument developed in [**12**], we show that, unfortunately, this is not the case.

THEOREM 2. *The system $\{e^{i\lambda_n t}\}$, where the λ_n are given by* (2), *is not a basis for $L^2(-\pi,\pi)$.*

The search for a conditional basis of complex exponentials continues.

2. Proof of Theorem 1. Suppose that the system $\{e^{i\lambda_n t}\}$ is an unconditional basis for $L^2(-\pi,\pi)$ and that the inequalities

$$A\sum |c_n|^2\|e^{i\lambda_n t}\|^2 \le \left\|\sum c_n e^{i\lambda_n t}\right\|^2 \le B\sum |c_n|^2\|e^{i\lambda_n t}\|^2 \tag{3}$$

hold for every convergent expansion $\sum c_n e^{i\lambda_n t}$. Let $\{\mu_n\}$ satisfy $|\lambda_n - \mu_n| \le L$. It is to be shown that if L is sufficiently small, then

$$\left\|\sum c_n(e^{i\lambda_n t} - e^{i\mu_n t})\right\| \le \theta \left\|\sum c_n e^{i\lambda_n t}\right\| \tag{4}$$

where $0 \le \theta < 1$ and $\{c_n\}$ is any *finite* sequence of scalars. This simple criterion ensures that the mapping $e^{i\lambda_n t} \to e^{i\mu_n t}$ can be extended to a bounded linear operator T on all of $L^2(-\pi,\pi)$ and that $\|I - T\| \le \theta < 1$ (see, e.g., [**9**, p. 84]). Thus T is an isomorphism and $\{e^{i\mu_n t}\}$ is a basis for $L^2(-\pi,\pi)$ equivalent to $\{e^{i\lambda_n t}\}$.

To establish (4) we first write

$$e^{i\mu_n t} - e^{i\lambda_n t} = e^{i\lambda_n t}(e^{i(\mu_n - \lambda_n)t} - 1)$$

and then expand $e^{i\delta t}$ in an everywhere-convergent Taylor series. Inasmuch as $\|t^k g(t)\| \le \pi^k \|g(t)\|$, we have for every finite sequence $\{c_n\}$

$$\begin{aligned}
\left\|\sum_n c_n(e^{i\lambda_n t} - e^{i\mu_n t})\right\| &= \left\|\sum_n c_n \sum_{k=1}^{\infty} \frac{[i(\mu_n - \lambda_n)]^k}{k!} t^k e^{i\lambda_n t}\right\| \\
&\le \sum_{k=1}^{\infty} \frac{\pi^k}{k!} \left\|\sum_n c_n [i(\mu_n - \lambda_n)]^k e^{i\lambda_n t}\right\| \\
&\le \sum_{k=1}^{\infty} \frac{\pi^k}{k!} \left\{B \sum_n |c_n|^2 |\lambda_n - \mu_n|^{2k} \|e^{i\lambda_n t}\|^2\right\}^{1/2} \\
&\le \sum_{k=1}^{\infty} \frac{\pi^k}{k!} L^k \left\{B \sum_n |c_n|^2 \|e^{i\lambda_n t}\|^2\right\}^{1/2} \\
&\le \sum_{k=1}^{\infty} \frac{(\pi L)^k}{k!} \cdot \frac{B}{A} \left\|\sum_n c_n e^{i\lambda_n t}\right\| \\
&= \frac{B}{A}(e^{\pi L} - 1) \left\|\sum_n c_n e^{i\lambda_n t}\right\|.
\end{aligned}$$

Notice that we have made use of both inequalities in (3). Setting $\theta = (B/A)(e^{\pi L} - 1)$, we have $\theta < 1$ provided L is sufficiently small.

REMARK. The argument just given is similar to the one used by Duffin and Eachus [**2**] to show that $\{e^{i\mu_n t}\}$ is a Riesz basis for $L^2(-\pi,\pi)$ whenever $|\mu_n - n| \le L < (\log 2)/\pi$. In this case $\lambda_n = n$ and Parseval's identity shows that $A = B = 1$; therefore (4) holds with $\theta = e^{\pi L} - 1 < 1$. By expanding the function $e^{i\delta t}$, not as a Taylor series, but in terms of the orthonormal basis $\{1, \cos nt, \sin(n - \frac{1}{2})t\}_{n=1}^{\infty}$, Kadec [**5**] deduced in the same way that (4) holds with $\theta = 1 - \cos \pi L + \sin \pi L$, *provided every μ_n is real.* Clearly $L < \frac{1}{4}$ implies $\theta < 1$.

3. Proof of Theorem 2. We argue by contradiction. If the system were a basis, then we could write

$$\sin t = \sum c_n e^{i\lambda_n t} \quad \text{(in the mean)}. \tag{5}$$

To compute the c_n, we shall make use of the Paley-Wiener space P consisting of all entire functions of exponential type at most π that are square integrable on the real axis. The inner product of two functions F and G in P is, by definition,

$$(F, G) = \int_{-\infty}^{\infty} F(x)\overline{G(x)}\, dx.$$

By virtue of the Paley-Wiener theorem, the complex Fourier transform

$$f(t) \to \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-izt}\, dt$$

is an isometric isomorphism from $L^2(-\pi, \pi)$ onto all of P. The exponentials $e^{i\lambda_n t}$ are sent to the "reproducing functions"

$$K_n(z) = \frac{\sin \pi(z - \lambda_n)}{\pi(z - \lambda_n)},$$

which then constitute a basis for P. Let $\{g_n\}$ be biorthogonal to $\{K_n\}$ in P. If we write $\sin t = (e^{it} - e^{-it})/2i$ and then apply the Fourier transform to both sides of (5), the result is

$$\frac{1}{2i}\left[\frac{\sin \pi(z-1)}{\pi(z-1)} - \frac{\sin \pi(z+1)}{\pi(z+1)}\right] = \sum c_n K_n(z).$$

Taking the inner product of each side with g_n, and remembering that

$$[\sin \pi(z - \overline{w})]/\pi(z - \overline{w})$$

is the reproducing kernel for P, we find

$$c_n = \frac{1}{2i}[\overline{g_n(1)} - \overline{g_n(-1)}].$$

Now, the g_n can be determined explicitly. Let $F(z)$ be the canonical product with zeros λ_n:

$$F(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right).$$

It was shown by Levinson [**6**, p. 11] that

$$F(z) = \int_{-\pi}^{\pi} f(t) e^{izt}\, dt,$$

where $f(t)$ is a multiple of $(\cos \frac{1}{2}t)^{-1/2} \sin \frac{1}{2}t$. Since $f(t)$ is integrable over $(-\pi, \pi)$, it follows that $F(z)$ is bounded along the real axis, and therefore each of the functions

$$F_n(z) = \frac{F(z)}{F'(\lambda_n)(z - \lambda_n)}$$

belongs to P. Since $(F_n, K_m) = F_n(\lambda_m) = \delta_{mn}$, it follows that the system $\{F_n\}$ is also biorthogonal to $\{K_n\}$ in P. But $\{K_n\}$ is *complete* and so has a unique biorthogonal sequence. Therefore, $F_n = g_n$ for every n, and we find

$$\begin{aligned} c_n &= \frac{1}{2i}[F_n(1) - F_n(-1)] = \frac{F(1)}{2iF'(\lambda_n)}\left[\frac{1}{1-\lambda_n} - \frac{1}{1+\lambda_n}\right] \\ &= \frac{iF(1)}{F'(\lambda_n)}\left[\frac{\lambda_n}{\lambda_n^2 - 1}\right]. \end{aligned}$$

Notice that $c_0 = 0$ and, since $F'(z)$ is *even*, $c_{-n} = -c_n$ for every n. Accordingly, (5) becomes

(6) $$\sin t = -2F(1)\sum_{n=1}^{\infty}\frac{\sin\lambda_n t}{\lambda_n F'(\lambda_n)}\left[\frac{\lambda_n^2}{\lambda_n^2-1}\right].$$

Now the values of $F'(\lambda_n)$ were determined explicitly in [**8**]:

$$F'(\lambda_n) = (-1)^n\lambda_n\Gamma^2\left(\frac{5}{4}\right)\frac{\Gamma(n)}{\Gamma(n+\frac{3}{2})} \qquad (n = 1, 2, \ldots).$$

Using the asymptotic formula $\Gamma(n)/\Gamma(n+\frac{3}{2}) = n^{-3/2} + O(n^{-5/2})$ [**1**], we have

(7) $$F'(\lambda_n) = A(-1)^n\lambda_n^{-1/2} + \varepsilon_n,$$

where $\varepsilon_n = O(1/n^{3/2})$ as $n \to \infty$. Since $|\lambda_n F'(\lambda_n)|$ tends to infinity with n, we see that the series in (6) differs from

(8) $$\sum_{n=1}^{\infty}\frac{\sin\lambda_n t}{\lambda_n F'(\lambda_n)}$$

by a series which converges uniformly on $[-\pi,\pi]$, and therefore our task is to show that (8) diverges in $L^2(-\pi,\pi)$. But if we apply the asymptotic formula (7) once again, we see that the difference between (8) and

(9) $$\frac{1}{A}\sum_{n=1}^{\infty}\frac{(-1)^n\sin\lambda_n t}{\sqrt{\lambda_n}}$$

is also uniformly convergent on $[-\pi,\pi]$. Accordingly, it suffices to show that (9) diverges in $L^2(-\pi,\pi)$.

Let $x = \pi - t$ $(0 \le t \le \pi)$. Then

$$\sum_{n=1}^{\infty}\frac{(-1)^n\sin\lambda_n t}{\sqrt{\lambda_n}} = \sum_{n=1}^{\infty}\frac{\sin(\frac{\pi}{4}-\lambda_n x)}{\sqrt{\lambda_n}}.$$

For $N = 1, 2, 3, \ldots$, let I_N be the interval $[0, \pi/32N]$. If $x \in I_N$ then $\pi/4 - \lambda_n x$ lies in the interval $[\pi/8, \pi/4]$ whenever $1 \le n \le 2N$, and hence, for these n, $\sin(\pi/4 - \lambda_n x) \ge B > 0$. Thus

$$\sum_{N}^{2N}\frac{\sin(\frac{\pi}{4}-\lambda_n x)}{\sqrt{\lambda_n}} \ge B\sum_{N}^{2N}\frac{1}{\sqrt{n+\frac{1}{4}}} \ge B\frac{N+1}{\sqrt{2N+\frac{1}{4}}} \ge C\sqrt{N},$$

where C is a positive constant independent of N. Accordingly

$$\left\| \sum_{N}^{2N} \frac{(-1)^n \sin \lambda_n t}{\sqrt{\lambda_n}} \right\|^2 \geq \left\| \sum_{N}^{2N} \frac{\sin(\frac{\pi}{4} - \lambda_n x)}{\sqrt{\lambda_n}} \right\|^2_{L^2(I_N)}$$

$$\geq \frac{1}{2\pi} C^2 N \left(\frac{\pi}{32N} \right) = \frac{C^2}{64}$$

for all N. This shows that the series in (5) does not converge in $L^2(-\pi, \pi)$ and hence that the system $\{e^{i\lambda_n t}\}$ fails to be a basis.

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