RESTRICTED LEFT INVERTIBLE TOEPLITZ OPERATORS
ON MULTIPLY CONNECTED DOMAINS

KEIJI IZUCHI AND SHÛICHI OHNO

Dedicated to Professor Junzo Wada on his sixtieth birthday

ABSTRACT. A characterization of restricted left invertible Toeplitz operators on multiply connected domains is given. To prove this, some extension theorems are given.

1. Introduction. Let Ω be a bounded connected open subset of the plane whose boundary Γ consists of finitely many disjoint analytic Jordan curves. We denote by $H^\infty(\Omega)$ the space of bounded holomorphic functions on Ω. In [1], Abrahamse studied Toeplitz operators on Ω and gave a left invertibility criterion for Toeplitz operators in a generalized sense. In [3], Clancey and Gosselin studied local properties of Toeplitz operators on the unit open disk and gave a characterization of restricted left invertible Toeplitz operators with the help of Younis’ extension theorem [7].

In this paper, we will extend the Clancey-Gosselin-Younis theorem to multiply connected domains (§3). To prove this, we need some extension theorems as in [7, 8]. Since $H^\infty(\Omega)$ is not a strongly logmodular algebra, here we cannot use the Younis extension theorems. We will give more general extension theorems (§2).

2. Extension theorems. Let A be a function algebra on X. Throughout this section, we assume that X is a totally disconnected compact Hausdorff space. A closed subset E of X will be called a peak set for A if there is a function f in A, which is called a peaking function for E, such that f = 1 on E and |f| < 1 on X\E. A subset S of X is called a weak peak set for A if it is an intersection of some peak sets. We consider the following separation condition.

For any open and closed subset U of X, there is a function G in A such that |G| > 1 on U and |G| < 1 on X\U.

The following theorem is a generalization of [8, Theorem 3.2 and Corollary 3.3].

THEOREM 1. Suppose that A satisfies separation condition (#). Let S be a weak peak set for A. If a function f in A satisfies |f| > 0 on S, then there exists a function F in A such that F = f on S and |F| > 0 on X.

PROOF. We may assume that f vanishes somewhere on X. Take an open and closed subset U of X such that S ⊆ U, |f| > 0 on U, and

\[ |f| \leq 1 \quad \text{on } X\setminus U. \]
Put $c = \inf\{|f(x)|; x \in U\}$. Then $c > 0$. Since $S$ is a weak peak set, there is a peak set $E$ for $A$ with $S \subseteq E \subseteq U$ [5, p. 56]. Let $h \in A$ be a peaking function for $E$. Replacing $h$ by high powers of $h$, we may assume

\[(2) \quad |h| < 1/2 \quad \text{on } X \setminus U.\]

By our separation condition, there is a function $G$ in $A$ such that

\[(3) \quad |G| \leq c/3 \quad \text{on } U \quad \text{and} \quad |G| \geq 3 \quad \text{on } X \setminus U.\]

Put $F = f + G(1 - h)$. Clearly $F \in A$ and $F = f$ on $S$. For $x \in U$, by (3) we have

\[|F(x)| \geq |f(x)| - |G(x)||1 - h(x)| \geq c/3 > 0.\]

For $x \in X \setminus U$, by (1), (2), and (3) we have

\[|F(x)| \geq |G(x)||1 - h(x)| - |f(x)| \geq 1/2 > 0.\]

Thus $|F| > 0$ on $X$.

**Remark 1.** Let $A$ be a strongly logmodular algebra on a totally disconnected compact space $X$, that is, $\log |A^{-1}| = C_R(X)$, where $A^{-1}$ denotes the set of invertible elements in $A$ and $C_R(X)$ denotes the space of real continuous functions on $X$. Then $A$ satisfies (#).

**Remark 2.** For a general function algebra $A$, the assertion of Theorem 1 is not true. Put $X = \{(z, t); z$ is a complex number and $t$ is a real number with $|z| = 1 - |t|$ and $|t| \leq 1\}$. Let $A = \{g \in C(X); \text{for each } t \text{ with } |t| < 1, \text{g}(z, t) \text{ has a continuous holomorphic extension to \{z \leq 1 - |t|\}\}$}. Then $S = \{(z, 0); |z| = 1\}$ is a peak set for $A$. Put $f(z, t) = (1 - |t|)z$. Then $f \in A$ and $|f| > 0$ on $S$, but it is easy to see that there is no $F \in A$ with $F = f$ on $S$ and $|F| > 0$ on $X$.

For a weak peak set $S$, put $A_S = \{f \in C(X); f|S \in A|S\}$. Then $A_S$ is a closed subalgebra of $C(X)$. For $\psi \in C(X)$ and $B \subseteq C(X)$, put $d(\psi, B) = \inf\{||\psi - f||; f \in B\}$. For $\psi \in C(X)$ and $F \subseteq X$, put $||\psi||_F = \sup\{||\psi(x)||; x \in F\}$. The following theorem is a generalization of [8, Theorem 3.1].

**Theorem 2.** Suppose that $A$ satisfies separation condition (#). Let $S$ be a weak peak set for $A$, and let $u$ be a function in $C(X)$ such that $\|u\| = 1$, $|u| = 1$ on $S$, and $d(u, A_S) < 1$. Then there is a function $\tilde{u}$ in $C(X)$ such that $|\tilde{u}| = |u|$ on $X$, $\tilde{u} = u$ on $S$, and $d(\tilde{u}, A) < 1$.

**Proof.** Since $d(u, A_S) < 1$, there exists a function $f$ in $A$ such that $\|u - f\|_S < 1$. Since $|u| = 1$ on $S$, $|f| > 0$ on $S$. By Theorem 1, there is a function $F$ in $A$ such that $F = f$ on $S$, and

\[(1) \quad |F| > 0 \quad \text{on } X.\]

Since $\|u - F\|_S = \|u - f\|_S < 1$, there is an open and closed subset $U$ of $X$ such that $S \subseteq U$ and

\[(2) \quad \|u - F\|_U < 1.\]

Since $S$ is a weak peak set, there is a peak set $E$ for $A$ such that $S \subseteq E \subseteq U$. Let $h \in A$ be a peaking function for $E$. Replacing $h$ by high powers of $(1 + h)/2$, we may assume that

\[(3) \quad |h| > 0 \quad \text{on } X \quad \text{and} \quad 0 < |hF| < 1 \quad \text{on } X \setminus U.\]
Put
\[ \tilde{u} = \begin{cases} \frac{uh |h|}{|h|} & \text{on } U, \\ |uh| F / |hF| & \text{on } X \setminus U. \end{cases} \]

By (3), \( \tilde{u} \in C(X) \). Clearly \( |\tilde{u}| = |u| \) on \( X \) and \( \tilde{u} = u \) on \( S \). To prove \( d(\tilde{u}, A) < 1 \), it is sufficient to prove \( \|\tilde{u} - hF\| < 1 \), because \( hF \in A \). We have
\[
\|\tilde{u} - hF\|_{X \setminus U} = \left\| \frac{|uh| F / |hF|}{|hF|} - \frac{hF |hF|}{|hF|} \right\|_{X \setminus U}
\leq \| |u| - |hF| \|_{X \setminus U}
< 1 \quad \text{by (3) and } \|u\| \leq 1.
\]

Also we have
\[
\|\tilde{u} - hF\|_U = \left\| \frac{uh}{|h|} - \frac{hF |h|}{|h|} \right\| = \|u - hF|h|\|_U.
\]

Let \( x \in U \). Then
\[
\|u(x) - F(x)|h(x)|\| = \| (1 - |h(x)|)u(x) + |h(x)|(u(x) - F(x)) \|
\leq 1 - |h(x)| + |h(x)| \|u - F\|_U \quad \text{by } \|u\| \leq 1, \|h\| \leq 1
= 1 - |h(x)| \{1 - \|u - F\|_U\}
< 1 \quad \text{by (2)}.
\]

Thus we get \( \|\tilde{u} - hF\|_U < 1 \), hence \( \|\tilde{u} - hF\| < 1 \).

If \( A \) is a strongly logmodular algebra, then \( A_S \), where \( S \) is a weak peak set for \( A \), is generated by \( A \) and \( \{ f^{-1}; f \in A \cap A_S^{-1} \} \) [2]. Younis used this property to prove his theorem [8, Theorem 3.1]. So the following theorem is another generalization of [8, Theorem 3.1].

**Theorem 3.** Suppose that \( A \) satisfies separation condition (\#). Let \( S \) be a weak peak set for \( A \) such that \( A_S \) is generated by \( A \) and \( \{ f^{-1}; f \in A \cap A_S^{-1} \} \). If \( u \) is a function in \( C(X) \) with \( \|u\| \leq 1 \) and \( d(u, A_S) < 1 \), then there exists a function \( \tilde{u} \) in \( C(X) \) such that \( |\tilde{u}| = |u| \) on \( X \), \( \tilde{u} = u \) on \( S \), and \( d(\tilde{u}, A) < 1 \).

**Note.** We do not assume \( |u| = 1 \) on \( S \).

**Proof.** By our assumption, there exist functions \( g \) in \( A \) and \( f \) in \( A \cap A_S^{-1} \) such that
\[
\|u - f^{-1} g\| < 1.
\]

Since \( f^{-1} \in A_S, |f^{-1}| > 0 \) on \( X \). By Theorem 1, there is a function \( F \) in \( A \) such that \( F = f^{-1} \) on \( S \) and \( |F| > 0 \) on \( X \). Put \( \tilde{u} = uF / |fF| \). Since \( |fF| > 0 \) on \( X \), \( \tilde{u} \in C(X) \). Clearly \( |\tilde{u}| = |u| \) on \( X \) and \( \tilde{u} = u \) on \( S \). To prove \( d(\tilde{u}, A) < 1 \), suppose not. Since \( \|\tilde{u}\| = \|u\| \leq 1 \), we have \( d(\tilde{u}, A) = 1 \). We note that the space of bounded linear functionals of \( C(X)/A \) may be identified with \( A^\perp \), the set of regular Borel measures \( \mu \) on \( X \) such that \( \int_X \phi d\mu = 0 \) for every \( \phi \in A \). Hence there exists \( \mu \in A^\perp \) with the unit total variation, \( \|\mu\| = 1 \), such that
\[
1 = \int_X \tilde{u} d\mu = \int_X uF / |fF| d\mu.
\]

Since \( \|u\| \leq 1 \), we get \( (uF / |fF|)\mu = |\mu| \), hence
\[
(2) \quad uF\mu = |fF| |\mu|.
\]
Then

\[
1 > \left| \int_X (u - f^{-1}g) \frac{ff^*}{\|ff^*\|} \, d\mu \right| \quad \text{by (1)}
\]

\[
= \frac{1}{\|ff^*\|} \left| \int_X uf^* \, d\mu \right| \quad \text{by } \mu \in A^-
\]

\[
= \frac{1}{\|ff^*\|} \left| \int_X |ff^*| \, d\mu \right| \quad \text{by (2)}
\]

\[
= 1.
\]

This contradiction shows \( d(\tilde{u}, A) < 1 \).

### 3. Restricted left invertible Toeplitz operators.

Let \( \Omega \) be a bounded connected open subset of the plane whose boundary \( \Gamma \) consists of finitely many disjoint analytic Jordan curves. Identifying a function in \( H^\infty(\Omega) \) with its boundary function, we may regard \( H^\infty(\Omega) \) as an essentially supremum norm closed subalgebra of \( L^\infty(m) \), where \( m \) is the arc length measure on \( \Gamma \). A closed subspace \( M \) of \( L^2(m) \) with \( M \neq \{0\} \) is called invariant if \( H^\infty(\Omega)M \subset M \), and it is called reducing if \( H^\infty(\Omega)M \subset M \) and \( \overline{H^\infty(\Omega)}M \subset M \). An invariant subspace of \( L^2(m) \) is called simple if it contains no reducing subspaces. For a closed subspace \( M \), \( P_M \) denotes the orthogonal projection of \( L^2(m) \) onto \( M \). For a function \( \psi \) in \( L^\infty(m) \), put \( T^M_\psi(f) = P_M(\psi f) \) for every \( f \) in \( M \). \( \mathcal{T}_\psi = \{T^M_\psi; M \text{ is a simply invariant subspace}\} \) is called a family of Toeplitz operators with a symbol \( \psi \). In the case that \( \Omega \) is the unit open disk, by Beurling's theorem, \( T^M_\psi \) is unitarily equivalent to the usual Toeplitz operator \( T_\psi \). We call \( T_\psi \) left invertible if \( T^M_\psi \) is a left invertible operator on \( M \) for every simply invariant subspace \( M \). In [1, Theorem 4.1], Abrahamse proved the following theorem which is a generalization of the Devinatz-Rabindranathan theorem (see [6, p. 119]).

**Theorem 4.** Let \( \psi \) be a function in \( L^\infty(m) \) with \( |\psi| = 1 \) a.e. \( dm \). Then the following conditions are equivalent:

(i) \( T_\psi \) is left invertible.

(ii) \( d(\psi, H^\infty(\Omega)) < 1 \).

We note that by his proof, (i) \( \Rightarrow \) (ii) is true for every \( \psi \in L^\infty(m) \) with \( ||\psi|| \leq 1 \).

Let \( X \) be the maximal ideal space of \( L^\infty(m) \). Then \( X \) is a totally disconnected compact Hausdorff space [4, p. 190]. Identifying a function in \( L^\infty(m) \) with its Gelfand transform, we have \( L^\infty(m) = C(X) \). We may consider \( H^\infty(\Omega) \) as a function algebra on \( X \) [4, p. 123]. Let \( S \) be a weak peak subset of \( X \) for \( H^\infty(\Omega) \). A family of Toeplitz operators \( \mathcal{T}_\psi, \psi \in C(X) \), is called \( S \)-restricted left invertible if there is a function \( \Psi \) in \( C(X) \) such that \( \Psi = \psi \) on \( S \) and \( T_\psi \) is left invertible. In the case that \( \Omega \) is the open unit disk, Clancey-Gosselin-Younis [3, 7] gave a characterization of \( S \)-restricted left invertible Toeplitz operators as follows: If \( |\psi| = 1 \) a.e. \( dm \), then the Toeplitz operator \( T_\psi \) is \( S \)-restricted left invertible if and only if \( d(\psi, H^\infty_S) < 1 \). We shall give a generalization of the above theorem to multiply connected domains as an application of §2.
THEOREM 5. Let $S$ be a weak peak subset of $X$ for $H^\infty(\Omega)$. Let $\psi$ be a function in $C(X)$ with $|\psi| = 1$ on $S$. Then the following conditions are equivalent:

(i) $T_\psi$ is $S$-restricted left invertible.

(ii) $d(\psi, H^\infty(\Omega)_S) < 1$.

PROOF. (ii) $\Rightarrow$ (i) Since $X$ is totally disconnected, we may assume that $|\psi| = 1$ on $X$. We note that $A = H^\infty(\Omega)$ satisfies separation condition (\#) in §2 [5, p. 119]. Then $H^\infty(\Omega)$, $\psi$, and $S$ satisfy all assumption of Theorem 2. Hence there exists a function $\Psi$ in $C(X)$ such that $|\Psi| = 1$ on $X$, $\Psi = \psi$ on $S$, and $d(\Psi, H^\infty(\Omega)) < 1$. By Theorem 4, $T_\Psi$ is left invertible. So (i) holds.

(i) $\Rightarrow$ (ii) Suppose that $T_\psi$ is $S$-restricted left invertible. By our definition, there is a function $\Psi$ in $C(X)$ such that $\Psi = \psi$ on $S$ and $T_\psi$ is left invertible. First we shall show that there is a function $h$ in $H^\infty(\Omega)$ such that

\begin{equation}
    h = 1 \text{ on } S, \quad ||\Psi h|| \leq 1, \quad \text{and} \quad |h| > 0 \text{ on } X.
\end{equation}

To prove this, put

\begin{equation}
    \phi = \max\{|\Psi|, 1\}.
\end{equation}

Then $\phi \in C(X)$, $\phi \geq 1$ on $X$, and $\phi = 1$ on $S$. By [5, p. 58], there is a function $g$ in $H^\infty(\Omega)$ such that $g = 1$ on $S$ and

\begin{equation}
    |g| \leq 1/\phi \text{ on } X.
\end{equation}

Then $||g|| = 1$. Take a positive integer $N$ with

\begin{equation}
    N \geq 4 \quad \text{and} \quad ||\Psi||(5/6)^N \leq 1.
\end{equation}

Put $h = ((g + 2)/3)^N$. Then $h \in H^\infty(\Omega)$, $h = 1$ on $S$, and $|h| > 0$ on $X$. If $x \in X$ with $|g(x)| \leq 1/2$, then

\begin{equation}
    |\Psi(x)h(x)| \leq ||\Psi||(5/6)^N \leq 1 \text{ by (4)}.
\end{equation}

If $x \in X$ with $1/2 \leq |g(x)| \leq 1$, then

\begin{equation}
    |\Psi(x)h(x)| \leq ((|g(x)| + 2)/3)^N |\phi(x)| \text{ by (2)}
\end{equation}

\begin{equation}
    \leq ((|g(x)| + 2)/3)^N/|g(x)| \text{ by (3)}
\end{equation}

\begin{equation}
    \leq 1.
\end{equation}

The last inequality follows from $((t + 2)/3)^N \leq t$ for $1/2 \leq t \leq 1$ and $N \geq 4$. Thus $||\Psi h|| \leq 1$ and we get (1).

We put $\Phi = \Psi h \in L^\infty(m)$. By (1),

\begin{equation}
    ||\Phi|| \leq 1 \quad \text{and} \quad \Phi = \psi \text{ on } S.
\end{equation}

Also we have that $T_\Phi$ is left invertible. To see this, let $M$ be a simply invariant subspace of $L^2(m)$. Since $h \in H^\infty(\Omega)$ and $|h| > 0$ on $X$, $T^M_h$ is left invertible. Since $T_\psi$ is left invertible, $T^{M}_\psi = T^M_\psi T^M_h$ is left invertible, so $T_\Phi$ is left invertible. As noted after Theorem 4, we get $d(\Phi, H^\infty(\Omega)) < ||\Phi|| \leq 1$. Hence

\begin{equation}
    d(\psi, H^\infty(\Omega)_S) = d(\Phi, H^\infty(\Omega)_S) \text{ by (5)}
\end{equation}

\begin{equation}
    \leq d(\Phi, H^\infty(\Omega)) < 1.
\end{equation}

This completes the proof.
References


Department of Mathematics, Kanagawa University, Yokohama 221, Japan