SOME FUNCTIONAL EQUATIONS IN BANACH ALGEBRAS
AND AN APPLICATION

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ABSTRACT. In this paper some results concerning certain functional equations in complex Banach algebras are presented. One of these results is used to prove an abstract generalization of the classical Jordan-Neumann characterization of pre-Hilbert space.

This paper is a continuation of our earlier work [6, 7]. Throughout this paper all Banach algebras and vector spaces are over the complex field \( C \). Our terminology and notation will be the same as in [7]. For Banach algebras and Banach *-algebras we refer to [1 and 4]. Our first few results characterize some additive functions.

**THEOREM 1.** Let \( A \) be a Banach *-algebra with identity \( e \). Let \( \lambda \) and \( \mu \) be automorphisms or antiautomorphisms (i.e. \( \lambda(ab) = \lambda(b)\lambda(a) \)) of \( A \) (any combination is allowed). If \( f : A \to A \) is an additive function such that \( f(a) = \lambda(a)f(a^{-1})\mu(a) \) for all normal invertible elements \( a \) of \( A \), then \( 2f(b) = \lambda(b)f(e) + f(e)\mu(b) \) for all \( b \in A \).

**PROOF.** Let us first assume that for the function \( f \)

\[
(1) \quad f(e) = 0
\]

holds and let us prove that in this case

\[
(2) \quad f(a) = 0
\]

is fulfilled for all \( a \in A \). Since \( f \) is by the assumption additive, (2) will be proved if we prove that (2) holds for all normal elements. Therefore let \( a \in A \) be an arbitrary normal element. One can choose rational numbers \( p \) and \( q \) such that \( a^{-1} \) and \( (e - b)^{-1} \) exist, where \( b = pe + qa \). Hence \( f(a) = 0 \) will be proved if we prove that \( f(b) = 0 \). Now according to the requirements of the theorem and (1) we have

\[
\begin{align*}
f(b) &= \lambda(b)f(b^{-1})\mu(b) = \lambda(b)f(b^{-1}(e - b))\mu(b) \\
&= \lambda(b)\lambda(b^{-1}(e - b))f(e - b^{-1}b)\mu(b^{-1}(e - b))\mu(b) \\
&= \lambda(e - b)f((e - b)^{-1} - e)\mu(e - b) \\
&= \lambda(e - b)\lambda((e - b)^{-1})f(e - b)\mu((e - b)^{-1})\mu(e - b) \\
&= -f(b).
\end{align*}
\]

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Hence \( f(b) = -f(b) \) which implies \( f(a) = 0 \) for an arbitrary normal element \( a \) and so (2) is proved. Let us now prove the theorem in full generality. For this purpose we introduce a function \( g: A \to A \) defined by

\[
g(a) = f(a) - \frac{1}{2} (\lambda(a)f(e) + f(e)\mu(a)),
\]

where \( f, \lambda, \) and \( \mu \) are such that the requirements of the theorem are fulfilled. It is obvious that \( g \) is additive. A simple calculation shows that \( g(a) = \lambda(a)g(a^{-1})\mu(a) \) holds for all normal invertible elements \( a \in A \). Since \( g(e) = 0 \) we have \( g(a) = 0 \) for all \( a \in A \). In other words

\[
f(a) = \frac{1}{2} (\lambda(a)f(e) + f(e)\mu(a)).
\]

The proof of the theorem is complete.

Using a similar approach as in the proof of Theorem 1 one can prove the following result.

**THEOREM 2.** Let \( A \) be a Banach \(*\)-algebra with identity \( e \). Let \( \lambda \) and \( \mu \) be automorphisms or antiautomorphisms of \( A \) (any combination is allowed) such that \( \lambda(a)\mu(a) = \mu(a)\lambda(a) \) for all normal invertible elements \( a \) of \( A \). If \( f: A \to A \) is an additive function such that \( f(a) = \lambda(a)\mu(a)fa^{-1} \) for all normal invertible elements \( a \) of \( A \), then \( 2f(b) = \lambda(b)f(e) + f(e)\mu(b) \) holds for all \( b \in A \).

One can now state two theorems in the spirit of Theorem 1 and Theorem 2 for arbitrary Banach algebras. More precisely, using a similar approach as in the proof of Theorem 1 one can prove the following two results.

**THEOREM 3.** Let \( A \) be a Banach algebra with identity \( e \). Let \( \lambda \) and \( \mu \) be automorphisms or antiautomorphisms (any combination is allowed). If \( f: A \to A \) is an additive function such that \( f(a) = \lambda(a)f(a^{-1})\mu(a) \) for all invertible elements \( a \) of \( A \), then \( 2f(b) = \lambda(b)f(e) + f(e)\mu(b) \) holds for all \( b \in A \).

**THEOREM 4.** Let \( A \) be a Banach algebra with identity \( e \). Let \( \lambda \) and \( \mu \) be automorphisms or antiautomorphisms (any combination is allowed) such that \( \lambda(a)\mu(a) = \mu(a)\lambda(a) \) for all invertible elements \( a \) of \( A \). If \( f: A \to A \) is an additive function such that \( f(a) = \lambda(a)\mu(a)fa^{-1} \) for all invertible elements \( a \) of \( A \), then \( 2f(b) = \lambda(b)f(e) + f(e)\mu(b) \) holds for all \( b \in A \).

The following corollaries are immediate consequences of the theorems above (by appropriately choosing \( \lambda \) and \( \mu \)).

**COROLLARY 1.** Let \( A \) be a Banach \(*\)-algebra with identity \( e \) and let \( f: A \to A \) be an additive function. The following statements are fulfilled.

1° If \( f(a) = af(a^{-1})a^* \) holds for all normal invertible elements \( a \) of \( A \), then \( f \) is of the form \( 2f(b) = bf(e) + f(e)b^* \).

2° If \( f(a) = a^*af(a^{-1})a \) holds for all normal invertible elements \( a \) of \( A \), then \( f \) is of the form \( 2f(b) = bf(e) + f(e)b \).

3° If \( f(a) = a^2f(a^{-1}) \) holds for all normal invertible elements \( a \) of \( A \), then \( f \) is of the form \( f(b) = bf(e) \).

4° If \( f(a) = a^*af(a^{-1}) \) holds for all normal invertible elements \( a \) of \( A \), then \( f \) is of the form \( f(b) = hf(e) \), where \( b = h + ik \) and \( h \) and \( k \) are hermitian.

**COROLLARY 2.** Let \( A \) be a Banach algebra with identity \( e \) and let \( f: A \to A \) be an additive function. The following statements are fulfilled.
1° If \( f(a) = af(a^{-1})a \) holds for all invertible elements \( a \) of \( A \), then \( f \) is of the form \( 2f(b) = bf(e) + f(e)b \).

2° If \( f(a) = a^2f(a^{-1}) \) holds for all invertible elements \( a \) of \( A \), then \( f \) is of the form \( f(b) = bf(e) \).

It should be mentioned that the results included in the corollaries above have been proved in our earlier paper [7] under a somewhat more complicated approach and a much stronger assumption that \( A \) is a hermitian Banach \(*\)-algebra (that is each hermitian element has real spectrum).

In the continuation we present the following results.

**THEOREM 5.** Let \( A \) be a Banach \(*\)-algebra with identity \( e \) and let \( f : A \to A \) be an additive function. Then the following statements are fulfilled.

1° If \( f(a) = -af(a^{-1})a^* \) holds for all normal invertible elements \( a \) of \( A \), then \( f \) is of the form \( 2i f(b) = b f(e) - f(ie)b^* \).

2° If \( f(a) = -a^*af(a^{-1}) \) holds for all normal invertible elements \( a \) of \( A \), then \( f \) is of the form \( f(b) = kf(ie) \), where \( b = h + ik \) and \( h \) and \( k \) are hermitian.

**PROOF.** If we introduce a function \( g : A \to A \) by the relation \( g(a) = f(ia) \), then 1° follows from statement 1° of Corollary 1 and 2° from statement 4° of Corollary 1.

We conclude our discussion of additive functions with the following results.

**THEOREM 6.** Let \( A \) be a Banach \(*\)-algebra with identity \( e \) and let \( f : A \to A \), \( g : A \to A \) be additive functions. Then the following statements are fulfilled.

1° If \( f(a) = ag(a^{-1})a^* \) holds for all normal invertible elements \( a \) of \( A \), then \( f \) and \( g \) are of the form

\[
2f(b) = b(f(e) - if(ie)) + (f(e) + if(ie))b^*,
\]

\[
2g(b) = b(f(e) + if(ie)) + (f(e) - if(ie))b^*.
\]

2° If \( f(a) = a^*ag(a^{-1}) \) holds for all normal invertible elements \( a \) of \( A \), then \( f \) and \( g \) are of the form

\[
f(b) = hf(e) + kf(ie), \quad g(b) = hf(e) - kf(ie),
\]

where \( b = h + ik \) and \( h \) and \( k \) are hermitian.

**PROOF.** Let us prove 1°. From

\[
f(a) = ag(a^{-1})a^*
\]

we obtain that also

\[
g(a) = af(a^{-1})a^*
\]

for all normal invertible \( a \in A \). Let us introduce \( F \) and \( G \) by \( F(a) = f(a) + g(a) \), \( G(a) = f(a) - g(a) \). \( F \) and \( G \) are obviously additive and from (3) and (4) one obtains easily that \( F(a) = aF(a^{-1})a^* \) and that \( G(a) = -aG(a^{-1})a^* \) holds for all normal invertible \( a \in A \). Hence according to statement 1° of Corollary 1 and statement 1° of Theorem 5 we have

\[
2F(a) = aF(e) + F(e)a^*, \quad 2iG(a) = aG(ie) - G(ie)a^*
\]

for all \( a \in A \). Since \( g(e) = f(e) \) and \( g(ie) = -f(ie) \), it follows that \( f(a) + g(a) = af(e) + f(e)a^* \) and \( f(a) - g(a) = -af(ie) + if(ie)a^* \) for all \( a \in A \). This proves
Similarly one can prove that 2° follows from statement 4° of Corollary 1 and statement 2° of Theorem 5.

This completes our discussion of additive functions. One may also use statement 1° of Corollary 1 to generalize the well-known Jordan-Neumann characterization of pre-Hilbert space. The rest of our paper does this.

**Theorem 7.** Let $A$ be a Banach $*$-algebra with identity $e$ and let $X$ be a vector space which is also a unitary left $A$-module. Suppose there exists a mapping $Q: X \to A$ with the properties

(i) $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$ for all pairs $x, y \in X$,

(ii) $Q(ax) = aQ(x)a^*$ for all $x \in X$ and all normal invertible $a$ of $A$.

Under these conditions for the mapping $B(\cdot, \cdot): X \times X \to A$ defined by the relation

$$B(x, y) = \tfrac{1}{4}(Q(x + y) - Q(x - y)) + \tfrac{i}{4}(Q(x + iy) - Q(x - iy))$$

the following statements are fulfilled.

1° $B(\cdot, \cdot)$ is additive in both arguments.

2° $B(ax, y) = aB(x, y)$, $B(x, ay) = B(x, y)a^*$ for all pairs $x, y \in X$ and all $a \in A$.

3° $Q(x) = B(x, x)$ for all $x \in X$.

The result above was proved in [7] under the stronger assumption that $A$ is a hermitian Banach $*$-algebra (see also [6]). If $A$ is the complex number field, then Theorem 7 reduces to S. Kurepa’s extension of Jordan-Neumann theorem (see [2, 3] and also [5]).

The proof of Theorem 7 we shall omit since it goes through in the same way as in [7] with the only exception that for the proof of statement 2° one has to use statement 1° of Corollary 1 instead of Theorem 1.3 in [7].

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**References**


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