A CLOSED SURFACE OF GENUS ONE IN $E^3$
CANNOT CONTAIN SEVEN CIRCLES THROUGH EACH POINT

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ABSTRACT. There exists a closed surface of genus one in $E^3$ which contains six circles through each point, but any closed surface of genus one in $E^3$ cannot contain seven circles through each point.

1. Introduction. A sphere in $E^3$ is characterized as a closed surface which contains an infinite number of circles through each point. But we do not know a surface other than a sphere or a plane, which contains many circles through each point of it.

In 1980, Richard Blum [1] found a closed $C^\infty$ surface of genus one which contains six circles through each point, and he gave a conjecture: A closed $C^\infty$ surface in $E^3$ which contains seven circles through each point is a sphere.

We proved in [3] that a closed simply connected $C^\infty$ surface in $E^3$ which contains three circles through each point is a sphere.

The purpose of this paper is to obtain the following theorem for a closed $C^\infty$ surface of genus one.

THEOREM. A closed $C^\infty$ surface of genus one in $E^3$ cannot contain seven circles through each point.

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2. Circles on a closed surface of genus one. Let $M$ be a closed surface of genus one. Then $M$ is topologically obtained from a square $ABCD$ by identifying $\overline{AB}$ with $\overline{DC}$ and $\overline{BC}$ with $\overline{AD}$.

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Let $\alpha$ and $\beta$ be the closed curves on $M$ corresponding to $AB$ and $BC$, respectively. Then the homotopy classes $[\alpha]$ and $[\beta]$ defined by $\alpha$ and $\beta$ are generators of the fundamental group $\pi_1(M)$ of $M$ (see the diagram). Then the intersection numbers are given as follows:

\[
\begin{align*}
\text{Int}(\alpha, \alpha) &= 0, \\
\text{Int}(\beta, \beta) &= 0, \\
\text{Int}(\alpha, \beta) &= 1 \quad \text{and} \quad \text{Int}(\beta, \alpha) = -1.
\end{align*}
\]

Let $\gamma$ be a closed curve on $M$. Then the homotopy class $[\gamma]$ defined by $\gamma$ can be written as $[\gamma] = m[\alpha] + n[\beta]$ for some integers $m$ and $n$.

The following facts are basic (see, for example, [4]).

**FACT 1.** $\gamma$ is a simple closed curve if and only if $m$ and $n$ are relatively prime.

**FACT 2.** $\gamma$ is knotted if and only if $|m| \geq 2$ and $|n| \geq 2$, or $M$ is knotted and $n \neq 0$.

We see from these facts that

(*) if $\gamma$ is a circle, then either $m = n = 0$ or $|m| = 1$ or $|n| = 1$.

**REMARK.** Since we disregard the orientation of curves, we identify $\gamma$ with $-\gamma$.

3. **Lemmas.** Let $M$ be a closed $C^\infty$ surface of genus one in $E^3$. Then we have the following lemmas for curves on $M$ in view of (*):

**LEMMA 1.** If two circles on $M$ are homotopic and if they have only one point in common, then they are tangent to each other at the point.

**PROOF.** Let $c_1$ and $c_2$ be two circles on $M$ which belong to a homotopy class $m[\alpha] + n[\beta]$. Since the intersection number $\text{Int}(\ , \ )$ is bilinear, $\text{Int}(c_1, c_2) = \text{Int}(ma + n\beta, ma + n\beta) = m^2 \text{Int}(\alpha, \alpha) + mn \text{Int}(\alpha, \beta) + nm \text{Int}(\beta, \alpha) + n^2 \text{Int}(\beta, \beta) = 0$.

Therefore $c_1$ and $c_2$ must be tangent to each other (cf. [4]).

**LEMMA 2.** Let $c_1 \in m[\alpha] + [\beta]$ and $c_2 \in [\alpha] + n[\beta]$. If $mn \geq 4$ or $mn \leq -2$, then at least one of $c_1$ and $c_2$ cannot be a circle.

**PROOF.** By the assumption, $\text{Int}(c_1, c_2) = mn - 1 \geq 3$ or $\leq -3$, so that $c_1$ and $c_2$ must have more than two points in common.

**LEMMA 3.** (1) If $|n' - n| = 2$, then two circles $c_1 \in [\alpha] + n[\beta]$ and $c_2 \in [\alpha] + n'[\beta]$ have two points in common.

(2) If $|n' - n| \geq 3$, then two curves $c_1 \in [\alpha] + n[\beta]$ and $c_2 \in [\alpha] + n'[\beta]$ must have more than two points in common, so that at least one of them cannot be a circle.

(3) Similar results hold for two curves $c_1 \in m[\alpha] + [\beta]$ and $c_2 \in m'[\alpha] + [\beta]$.

**PROOF.** (1) By the assumption, $\text{Int}(c_1, c_2) = n' - n = \pm 2$, so that $c_1$ and $c_2$ have two points in common.

(2) By the assumption, $\text{Int}(c_1, c_2) = n' - n \geq 3$ or $\leq -3$, so that $c_1$ and $c_2$ must have more than two points in common.

**LEMMA 4.** If two circles $c_1 \in 0 \cdot [\alpha] + 0 \cdot [\beta]$ and $c_2$ have only one point in common, then they must be tangent to each other at the point.

**PROOF.** $\text{Int}(c_1, c_2) = 0$, so that $c_1$ and $c_2$ must be tangent to each other at the point.
4. Proof of the theorem. Now we will prove our theorem by using the above lemmas and the following result:

Proposition [3]. Let $M$ be a $C^\infty$ surface in $E^3$. Suppose that, through each point of $M$, there exist three circles of $E^3$ contained in $M$, any two of which are tangent to each other or have two points in common. Then $M$ is (a part of) a sphere or a plane.

Let $p$ be an arbitrary point of $M$. Then, from Lemmas 2 and 3, we see that the maximal sets of homotopy classes which may contain circles through $p$ simultaneously are

$$\{@, [\alpha], m[\alpha] + [\beta], (m + 1)[\alpha] + [\beta], (m + 2)[\alpha] + [\beta]\}$$

for some integer $m$, or

$$\{@, [\beta], [\alpha] + n[\beta], [\alpha] + (n + 1)[\beta], [\alpha] + (n + 2)[\beta]\}$$

for some integer $n$, where @ stands for $0 \cdot [\alpha] + 0 \cdot [\beta]$.

It follows from Lemmas 1–4 that each of the above sets is divided into three subsets with respect to the property that any two circles through $p$ which belong to homotopy classes in one subset are tangent to each other or have two points in common:

$$\{@, [\alpha], m[\alpha] + [\beta], (m + 1)[\alpha] + [\beta], (m + 2)[\alpha] + [\beta]\}$$

$$= \{@, [\alpha]\} \cup \{m[\alpha] + [\beta], (m + 2)[\alpha] + [\beta]\} \cup \{(m + 1)[\alpha] + [\beta]\},$$

$$\{@, [\beta], [\alpha] + n[\beta], [\alpha] + (n + 1)[\beta], [\alpha] + (n + 2)[\beta]\}$$

$$= \{@, [\beta]\} \cup \{[\alpha] + n[\beta], [\alpha] + (n + 2)[\beta]\} \cup \{[\alpha] + (n + 1)[\beta]\}.$$  

Suppose there exist seven circles through $p$. Then at least one subset must contain three circles and any two of them are tangent to each other or have two points in common. Since $p$ is arbitrary, $M$ must be a sphere or a plane by the Proposition. This contradicts the fact that $M$ is of genus one. Q.E.D.

References


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