ABSTRACT. If \( G \) is a finite Abelian group acting as a \( \mathbb{Z}_p \)-homology \( n \)-sphere \( X \) (where \( P \) is the set of primes dividing \( |G| \)), then there is an integer valued function \( n(\cdot, G) \) defined on the prime power subgroups \( H \) of \( G \) such that \( X^H \) has the \( \mathbb{Z}_p \)-homology of a sphere \( S^{n(H,G)} \). We prove here that there exists a real representation \( R \) of \( G \) such that for any prime power subgroup \( H \) of \( G \),
\[
\dim(S(R^H)) = n(H, G)
\]
where \( S(R^H) \) is the unit sphere of \( R^H \), provided that \( n - n(H,G) \) is even whenever \( H \) is a 2-subgroup of \( G \).

0. Introduction. Suppose that \( G \) is a finite Abelian group and let \( P \) be the set of primes dividing \( |G| \). If \( G \) acts on a finite CW-complex \( X \) which has the \( \mathbb{Z}_p \)-homology of \( S^n \), then for any \( p \in P \) and \( p \)-subgroup \( H \) of \( G \) the fixed point set of \( H \) on \( X \), \( X^H \), has the \( \mathbb{Z}_p \)-homology of \( S^{n(H,G)} \) for some integer \( n(H,G) \geq -1 \) (less than or equal to 0 signifies empty). This is a well-known consequence of Smith theory (see [2, III or 1, IV], e.g.). Thus we obtain in this way an integer valued function, \( n(\cdot, G) \), defined on the set of \( p \)-subgroups of \( G \) by \( H \mapsto n(H, G) \) (note that \( n(e,G) = n \)). This function is called the "dimension function" and has a considerable literature (see [5, 3, 6, 7, 9]; [10] gives a related extensive bibliography).

The function \( n(\cdot, G) \) satisfies the following well-known conditions (see [1, XIII, 2.3; IV, 4.4, 4.7]):

1. (Borel Formula) If \( H \leq K \) are both \( p \)-subgroups of \( G \) and \( K/H = \mathbb{Z}_p + \mathbb{Z}_p \), then \( n(H,G) - n(K,G) = \sum (n(K'/G) - n(K,G)) \) with the sum over all \( H \leq K' \leq K \) such that \( K'/H = \mathbb{Z}_p \).
2. If \( H \leq K \) are \( p \)-subgroups of \( G \), then \( n(K,G) \leq n(H,G) \).
3. If \( H \leq K \) are \( p \)-subgroups of \( G \) with \( K/H = \mathbb{Z}_p \) and \( p \) odd, then \( n(H,G) - n(K,G) \) is even.
4. If \( H \leq K' \leq K \) are 2-subgroups of \( G \) such that \( K/H = \mathbb{Z}_4 \), \( K'/H = \mathbb{Z}_2 \), then \( n(H,G) - n(K',G) \) is even.

For each \( p \in P \), let \( G(p) \) denote the \( p \)-Sylow subgroup of \( G \) and set \( N(\cdot, G) = n(\cdot, G) + 1 \). The function \( N(\cdot, G) \) restricted to the subgroups of \( G(p) \) will naturally be denoted by \( N(\cdot, G(p)) \). In [8] it was shown that \( N(\cdot, G(p)) \) is realized by a real representation \( V(p) \) of \( G(p) \) which means that for each \( H \leq G(p) \),
\[
\dim V(p)^H = N(H, G(p)) = N(H,G).
\]
If \( S(V(p)) \) denotes the unit sphere of \( V(p) \), then \( \dim S(V(p)^H) = n(H,G) \).

Here we are interested in the existence of a real representation \( R \) of the Abelian group \( G \) such that for any \( p \)-subgroup \( H \) of \( G \) (for any \( p \in P \)), \( \dim R^H = N(H,G) \). Thus \( R \) would be a simultaneous realization of the functions \( N(\cdot, G(p)) \), \( p \in P \).
It should be noted that in the special case where $X^H$ is a homology sphere for all $H \leq G$, then $N(\ ,G)$ is defined on all subgroups of $G$ and tom Dieck has shown in [3] that $N(\ ,G)$ is realized by a difference of representations. In general, this is best possible.

Clearly $N(\ ,G)$ satisfies conditions 1–4 if $n(\ ,G)$ does. We will denote $N(e,G)$ by $N$. As an example to show that some condition beyond 1–4 is needed, suppose $G = \mathbb{Z}_6$, $N = 2$, $N(\mathbb{Z}_3,G) = 0$, and $N(\mathbb{Z}_2,G) = 1$. Then $N(\ ,G)$ satisfies conditions 1–4 but there is no real representation of $G$ which realizes these numbers simultaneously as dimensions. From now on we will assume the following “orientation preserving” condition holds, in addition to conditions 1–4:

5. If $H$ is any 2-subgroup of $G$, then $N - N(H, G)$ is even.

We obtain the following theorem and corollary:

**THEOREM.** Let $G$ be a finite Abelian group and suppose $N(\ ,G)$ is a nonnegative integer valued function defined on the $p$-subgroups of $G$ for all $p \mid |G|$, satisfying conditions 1–5. Then there exists a real representation $R$ of $G$ such that for any $p$-subgroup $H$ of $G$, $p \mid |G|$, $\dim R^H = N(H,G)$. Furthermore, if $R$ is another such representation of $G$ then for all subgroups $H$ of $G$,

$$\dim R^H \equiv \dim R^H \pmod{2}.$$ 

**COROLLARY.** Let $G$ be a finite Abelian group and suppose the 2-Sylow subgroup of $G$ is cyclic. If $N(\ ,G)$ is a nonnegative integer valued function defined on the $p$-subgroups of $G$, $p \mid |G|$ satisfying only conditions 1–4, then there exists a real representation $R$ such that for any $p$-subgroup $H$ of $G$, $p \mid |G|$, $\dim R^H = N(H,G) + 1$.

In §§1 and 2 we prove the theorem and corollary respectively. We thank the referee for several suggestions leading to an improved exposition.

1. **Proof of the theorem.** Let $G$ be a finite Abelian group and suppose $N(\ ,G)$ is a nonnegative integer valued function on the $p$-subgroups of $G$, for all $p \in P$, satisfying conditions 1–5. By [8], for each $p \in P$ there is a representation $V(p)$ of the $p$-Sylow subgroup $G(p)$ of $G$ such that $\dim V(p)^H = N(H,G(p))$ for all $H \leq G(p)$. Let $V = \bigotimes_{p \mid |G|} V(p)$. Then $V$ is a representation of $G$ and we will prove by induction on $|G|$ that $V$ contains a subrepresentation $R$ of $G$ of dimension $N = N(e,G)$ such that $\dim R^H = N(H,G)$ for all $p$-subgroups $H$ of $G$, all $p \in P$. So if $|K| < |G|$, $N(\ ,K)$ is a nonnegative integer valued function on the prime power subgroups of $K$ and $W(p)$ is a representation of $K(p)$ realizing $N(\ ,K(p))$, we can assume $W = \bigotimes_{p \mid |K|} W(p)$ contains a subrepresentation realizing $N(\ ,K)$.

Suppose that $N(\ ,G)$ is a nonnegative integer valued function defined on the prime power order subgroups of an Abelian group $G$ satisfying conditions 1–5 and suppose $N(e,G) = N(H,G)$ for some $H \leq G(p)$, $|H| = p$. Then for any prime power order subgroup $K$ of $G$, define $N(K/K \cap H, G/H) = N(K,G)$. It is clear that $N(\ ,G/H)$ satisfies conditions 1–5. Moreover, for any $K \leq G(p)$,

$$N(K,G) = N(K,G(p)) = N(KH,G(p)) = N(KH,G) = N(KH/H,G/H).$$

For by induction we can assume $N(K',G(p)) = N(K'H,G(p))$ for any $K' \not\leq K$ and clearly we can assume $H \not\leq K$. Select $K' < K$ such that $|K/K'| = p$ and use
condition 1 (Borel Formula) on $K' \leq KH$ to obtain $N(K, G(p)) = N(KH, G(p))$. It follows that in this case, a representation of $G/H$ realizing $N(, G/H)$ can be regarded as an unfaithful representation of $G$ (with kernel at least $H$) realizing $N(, G)$.

Now for each $p \in \mathcal{P}$ and each $Z_p \leq G$, we must have $N - N(Z_p, G) > 0$, otherwise by the observation above we could assume we are given a dimension function on $G/Z_p$. Of all the differences $N - N(Z_p, G)$, $p \in \mathcal{P}$, let $p_0$ and $H_0 = Z_{p_0}$ be such that $N - N(H_0, G)$ is a minimum. Then the representation $V(p_0)$ of $G(p_0)$ (the $p_0$-Sylow subgroup of $G$) contains an irreducible subrepresentation $W(p_0)$ of $G(p_0)$ on which $H_0$ acts without (nonzero) fixed points. For $q \neq p_0$ select $H = Z_q \leq G(q)$ such that $N - N(H, G)$ is least for $q$ (in general $N - N(H_0, G) \leq N - N(H, G)$) and let $W(q)$ be an irreducible subrepresentation of $V(q)$ on which $H$ acts without fixed points. Then $\mathcal{W} = \otimes_{q \mid |G|} W(q)$ is a $G$-subrepresentation of $\mathcal{V} = \otimes_{q \mid |G|} V(q)$.

Let $R_1$ be an irreducible $G$-subrepresentation of $\mathcal{W}$. If $|G|$ is larger than 2, $R_1$ has dimension 2, since $R_1$ induces a free, irreducible, real representation of a cyclic group of order larger than 2 (the cyclic group is $G$/kernel of $R_1 = $ kernel of $\mathcal{W}$).

Now $R_1$, is being a representation of $G$, has associated to it a dimension function $N_1(, G)$ defined on all subgroups $H$ of $G$ by $N_1(H, G) = \dim R_1^H$. Set $N_1(, G) = N(, G) - N_1(, G)$. It is easy to verify that $N_1(, G)$ is a dimension function defined on the prime power subgroups of $G$ satisfying conditions 1–5.

Since $N_1(e, G) < N(e, G)$ and $N_1(H_0, G) = N(H_0, G)$ we are presented with two situations: (a) $N_1(e, G) = N_1(H_0, G)$ or (b) $N_1(e, G) > N_1(H_0, G)$.

In case (a) the function $N_1(, G)$ may be replaced (as noted above) by a dimension function defined on the prime power subgroups of $G/H_0$. Since for any $q$-subgroup $K$ of $G$, $\dim R^K = \dim W(q)^K$, the subrepresentation $W(q)^\perp$ of $V(q)$ realizes the dimension function $N_1(, G(q))$. Since $|G/H_0| < |G|$ by induction the tensor product of all the $W(q)^\perp$ contains a subrepresentation $R^*$ of $G/H_0$ (which may be thought of as an unfaithful representation of $G$). $R^*$ is a $G$-subrepresentation of $\mathcal{V}$, the tensor product of all the $V(q)$. $R^* \oplus R_1$ is the required representation in this case.

In (b), where we have $N_1(e, G) > N_1(H_0, G)$, note that $N_1(e, G) - N_1(H_0, G)$ is still a minimum of all differences $N_1(e, G) - N_1(H, G)$. Since the function $N_1(, G(Q))$ is realized by the subrepresentation $W(q)^\perp$ of $V(q)$, we can repeat the procedure again obtaining another irreducible subrepresentation $R_2$ or $G$ with an associated dimension function $N_2(, G)$ defined on the prime power subgroups of $G$ (it is the restriction of a dimension function defined on all subgroups of $G$). Letting $N_2(, G) = N_1(, G) = N_2(, G)$ we again have a dimension function satisfying conditions 1–5 and we proceed as above. Eventually we obtain a dimension function $N_k(, G)$ such that $N_k(e, G) = N_k(H_0, G) (k = N - N(H_0, G))$. By case (a) and induction there is a $G$-subrepresentation of $\mathcal{V}$, $R^*$ realizing $N_k(, G)$. The representation $R = R^* \oplus R_1 \oplus R_2 \oplus \cdots \oplus R_k$ is the required $G$-subrepresentation of $\mathcal{V}$.

Now suppose $\bar{R}$ is another $G$-subrepresentation such that for any prime power order subgroup $H$ of $G$, $\dim \bar{R}^H = N(H, G)$. Let $K$ be an arbitrary subgroup of $G$ and by induction assume $\dim \bar{R}^K - \dim R^k$ is even for all subgroups $K$ of $G$ with $|K| < |H|$. Select $K \leq H$ so that $|H/K|$ is an odd prime $p$ (if this is not possible
then $H$ is a 2-group and $\dim \overline{R}_H^H - \dim R^H$ is zero). The group $H/K = \mathbb{Z}_p$ acts on both $\overline{R}_K^H$ and $R^K$. It follows that both $\dim \overline{R}_K^H - \dim \overline{R}^H$ and $\dim R^K - \dim R^H$ are even and therefore $\dim \overline{R}^H - \dim R^H$ is even. This completes the proof of the theorem. □

2. Proof of the corollary. Suppose $G$ is a finite Abelian group with cyclic 2-Sylow subgroup, $G(2)$, and suppose $N(\cdot, G)$ is a nonnegative integer valued function defined on the $p$-subgroups of $G$, $p \mid |G|$, satisfying conditions 1–4. By condition 4, for any proper subgroup $H$ of $G(2)$, $N - N(H, G)$ is even. Suppose that $N - N(G(2), G)$ is odd. For each $p \mid |G|$, let $\overline{V}(p) = V(p) \oplus 1$, where 1 denotes the trivial one-dimensional representation of $G(p)$. It is easy to see that $\overline{V}(p)$ realizes $N(\cdot, G(p)) + 1$. The function $N^*(\cdot, G) = N(\cdot, G) + 1$ corresponds to the $G$-action on the unreduced suspension on $X$.

Now since $N - N(G(2), G)$ is odd, $\overline{V}(2)$ has an irreducible summand of dimension 1 on which $H$, the maximal proper subgroup of $G(2)$, acts trivially and on which $G(2)$ acts nontrivially. Denote this summand by $W(2)$ for any $p \neq 2$ let $W(p)$ be a one-dimensional trivial subrepresentation of $\overline{V}(p)$. Then $R_1 \otimes_{p \mid |G|} W(p)$ is a 1-dimensional $G$-representation with a very large kernel and is a subrepresentation of $\overline{V} = \bigotimes_{p \mid |G|} \overline{V}(p)$. Let $N_1(\cdot, G)$ be the dimension function associated with $R_1$ ($N_1(\cdot, G)$ is actually defined on all subgroups of $G$). Setting $\overline{N}(\cdot, G) = N^*(\cdot, G) - N_1(\cdot, G)$ we see that $N - \overline{N}(H, G)$ is now even for all prime power subgroups of $G$ so $\overline{N}(\cdot, G)$ satisfies conditions 1–5. By the argument §1, $\overline{N}(\cdot, G)$ is realized by a subrepresentation $R$ of $\bigotimes_{p \mid |G|} W(P)^\perp$, since $\overline{N}(\cdot, G(p))$ is realized by $W(p)^\perp$ for all $p \mid |G|$. Then $R \oplus R_1$ is a subrepresentation of the $G$-representation $\overline{V}$ which realizes $N^*(\cdot, G)$. This establishes the corollary.

EXAMPLES. Let $G = \mathbb{Z}_6$, $N = 2$, $N(\mathbb{Z}_2, G) = 1$, $N(\mathbb{Z}_3, G) = 0$. If we “suspend” $N(\cdot, G)$ we have $N^* = 3$, $N^*(\mathbb{Z}_2, G) = 2$, $N^*(\mathbb{Z}_3, G) = 1$. Then the construction of §§2 and 1 yields the 3-dimensional representation of $G$, given on a generator by

$$
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & R(2\pi/3)
\end{pmatrix}
$$

where $R(2\pi/3)$ is a $2 \times 2$ rotation matrix.

REFERENCES


DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF MISSOURI–ST. LOUIS, ST. LOUIS, MISSOURI 63121