SOME HOMOTOPY PROPERTIES OF THE HOMEOMORPHISM GROUPS OF $R^\infty(Q^\infty)$-MANIFOLDS

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ABSTRACT. In this note we will prove that, given an $R^\infty(Q^\infty)$-manifold $M$, there is a deformation of $\text{Homo}(M)$ into $\text{Homeo}(M)$ whose final stage is a weak homotopy equivalence, and that if $M$ has the homotopy type of a finite simplicial complex, then $\text{Homeo}(M)$ is an ANE($C\mathcal{W}(M)$) and an ANE($C\mathcal{W}(\mathcal{C}), G_6$).

0. Introduction. All spaces are Hausdorff and maps are continuous. The composition of two maps $f: A \to B$ and $g: B \to C$ will be denoted by $gf$. Let $\mathcal{G}$ denote the category whose objects are compactly generated spaces ($k$-spaces) and whose morphisms are continuous maps [Gr, 8.1]. Given two spaces $X$ and $Y$, following [Gr], let $X \times Y$ denote the product of $X$ and $Y$ in $\mathcal{G}$, let $X \times_c Y$ be the Cartesian product, let $C(X,Y)$ be the space of maps from $X$ to $Y$ with the compact-open topology, and let $Y^X = kC(X,Y)$ [Gr, 8.14]. Similarly, let $\text{Homeo}(X)$ ($\text{Homo}(X)$, resp.) denote the space of homeomorphisms (homotopy equivalences, resp.) of $X$ with compact-open topology as a subspace of $C(X,X)$. Recall that $k\text{Homeo}(X)$ and $k\text{Homo}(X)$ are subspaces of $X \times X$ in the sense of [Hy, p. 201]. Following [L3], let $C\mathcal{W}(\mathcal{C})$ and $C\mathcal{W}(M)$ denote the classes of pseudo CW-complexes generated by the class $\mathcal{C}$ of Hausdorff compact spaces and by the class $\mathcal{M}$ of metric spaces, respectively. Let $E^\infty = \varinjlim E^n$, where $E^n$ is either the $n$-Euclidean space or the $n$-fold product of the Hilbert cube $Q$. By an $E^\infty$-manifold, we mean a separable paracom pact space which is locally homeomorphic to $E^\infty$.

In this note we will prove that given an $E^\infty$-manifold $M$, there is a deformation of $\text{Homo}(M)$ in itself into $\text{Homeo}(M)$ whose final stage is a weak homotopy equivalence (this partially answers a question raised by P. J. Kahn some time ago [L3, §0]), and that if $M$ has the homotopy type of a finite complex, then $\text{Homeo}(M)$ is an ANE($C\mathcal{W}(M)$) and is an ANE($C\mathcal{W}(\mathcal{C}), G_6$). By a space $S \in \text{ANE}(\mathcal{X}, G_6)$, we mean that if $X \in \mathcal{X}$ and $A$ is a closed $G_6$-subset of $X$, then every map $f: A \to S$ has a continuous extension over a neighborhood of $A$ in $X$.

Throughout this note $I = [0,1]$. By an $E^\infty$-manifold fiber bundle, we mean a locally trivial bundle in $\mathcal{G}$, $p: X \to B$, whose fiber is an $E^\infty$-manifold. If $(X,p,B)$ is a bundle, $W$ a subset of $X$, and $A$ a subset of $B$, we write $W_A = W \cap p^{-1}(A)$; specially, $X_A = p^{-1}(A)$. Let $(X,p,B)$ and $(Y,p',B)$ be bundles and $W$ a subset of $X$. A map $f: W \to Y$ is said to be fiber preserving (f.p.) if $p'f = pW$. If $f: W \to B \times Z$ is an f.p. map, for each $b \in B$ let $f_b: W_{(b)} \to Z$ denote the map $p_Z f|W_{(b)}$ where $p_Z: B \times Z \to Z$ is the projection. Given an $R^\infty(Q^\infty)$-manifold $M$,
following \([L_1, L_2]\), we write \(M = \text{dirlim } M_n\), where \(M_n\) is a compact submanifold (compact \(Q\)-submanifold) of \(M_{n+1}\) for each \(n = 1, 2, \ldots\).

Given a map \(f : B \to Y^X\) or \(f : B \to C(X, Y)\), by the *associate* of \(f\) we mean the f.p. function \(\hat{f} : B \times Y \to B \times Y\) defined by \(\hat{f}(b, x) = (b, f(b)(x))\), and vice versa; we also call \(f\) the associate of \(\hat{f}\) \([D, p. 261]\). By \([Gr, 8.17]\), the continuity of \(f\) and that of its associate are equivalent if \(B, X, Y \in \mathcal{CG}\). Given a subset \(A\) of \(X\), let \(i_A\) denote the inclusion \(A \to X\) and \(\text{id}_A\) the identity map on \(A\).

1. **A canonical deformation.** The main result of this section is Theorem 1.3 which proves the existence of a deformation of the space of homotopy equivalences into the space of homeomorphisms. The proof of the following lemma is straightforward.

**Lemma 1.1.** Let \(X, Y,\) and \(Z\) be topological spaces. Let

\[ F : C(X, Y) \to C(X \times_c Z, Y \times_c Z) \]

be the map defined by \(F(g) = g \times \text{id}_Z\). Then \(F\) is an embedding, i.e., \(C(X, Y)\) is homeomorphic to the subspace \(F(C(X, Y))\) of \(C(X \times_c Z, Y \times_c Z)\). \(\square\)

We need another lemma for the proof of Theorem 1.3. Recall that \(X \times_c I \cong X \times I\) for \(X \in \mathcal{CG}\) \([Gr, 8.11]\).

**Lemma 1.2.** Let \(M, N\) be topological spaces. If \(g : M \times_c I \to M\) and \(h : N \times_c I \to N\) are given homotopies, then \(g\) and \(h\) induce the homotopies \(G\) and \(H\) on \(C(M, N)\) defined by \(G(f, t) = f g_t\) and \(H(f, t) = h_t f\) for all \(f \in C(M, N)\) and \(t \in I\).

**Proof.** For the continuity of \(G\), consider the following composition:

\[
\overline{g} : C(M, N) \xrightarrow{\overline{p}_M} C(M \times_c I, N) \xrightarrow{\overline{g}} C(M \times_c I, N) \xrightarrow{\sigma} C(I, C(M, N)),
\]

where \(\overline{p}_M, \overline{g}\) are maps naturally induced \([D, p. 259]\) from the projections \(p_M : M \times_c I \to M\) and \(g' : M \times_c I \to M \times_c I\) defined by \(g'(x, t) = (g(x, t), t)\), and where \(\sigma\) is the embedding defined by \(\sigma(f)(t)(x) = f(x, t)\) for all \(t \in I\) and \(x \in M\) \([J, \text{Theorem 1.1}]\). Define

\[
G : C(M, N) \times_c I \xrightarrow{\overline{g} \times \text{id}_I} C(I, C(M, N)) \times_c I \xrightarrow{\sigma} C(M, N),
\]

where \(\sigma\) is the evaluation map. Then \(G\) is a homotopy. Let \(f \in C(M, N)\) and \(t \in I\). We have

\[
G(f, t) = e((\sigma g' \overline{p}_M)(f, t)) = e(\sigma f_p M g')(t) = e(\sigma f_{p M} g')(t) = \sigma(f_{p M} g')(t) = f_{p M} g'(\cdot, t) = f g_t.
\]

For the continuity of \(H\), consider the composition

\[
\overline{h} : C(M, N) \xrightarrow{F} C(M \times_c I, N \times_c I) \xrightarrow{\hat{h}} C(M \times_c I, N) \xrightarrow{\sigma} C(I, C(M, N)),
\]

where the embedding \(F\) is given by Lemma 1.1, \(\hat{h}\) is induced from \(h\) and \(\sigma\) is from \([J, \text{Theorem 1.1}]\) as above. Define

\[
H : C(M, N) \times_c I \xrightarrow{\overline{h} \times \text{id}_I} C(I, C(M, N)) \times_c I \xrightarrow{\sigma} C(M, N).
\]
Then $H$ is continuous. Finally, let $f \in C(M, N)$ and $t \in I$. We have

\[
H(f, t) = e(\overline{h} \times \text{id}_I)(f, t) = e(\overline{h}(f), t) = \overline{h}(f)(t)
\]

\[
= (\sigma \overline{h}F(f))(t) = \overline{h}(F(f))(\cdot, t)
\]

\[
= (\overline{h}(f \times \text{id}_I))(\cdot, t) = h((f \times \text{id}_I)(\cdot, t))
\]

\[
= h(f(\cdot), t) = htf. \quad \square
\]

**REMARK 1.** In Lemma 1.2, if $M$ and $N$ are $k$-spaces, then $G$ and $H$ are also homotopies on $N^M = kC(M, N)$ (by use of [Gr, 8.8(vi)]).

**THEOREM 1.3.** Let $M$ be an $E^\infty$-manifold. Then, there is a deformation of $\text{Homo}(M)$ in itself into $\text{Homeo}(M)$ whose final map $r : \text{Homo}(M) \rightarrow \text{Homeo}(M)$ is a weak homotopy equivalence.

**PROOF.** To avoid confusion, we let $N$ be another copy of $M$ and consider $\text{Homo}(M, N)$ and $\text{Homeo}(M, N)$ instead of $\text{Homo}(M)$ and $\text{Homeo}(M)$, respectively. By [He, Theorem C], we can identify $M$ and $N$ with $K \times E^\infty$ and $L \times E^\infty$, respectively, where $K$ and $L$ are locally finite simplicial complexes. We also identify $K$ and $L$ with $K \times 0$ and $L \times 0$, respectively. Let $g : M \times I \rightarrow M$ and $h : N \times I \rightarrow N$ be natural strong deformation retractions of $M$ and $N$ onto $K$ and $L$, respectively, with $g_1 = p_K$ and $h_1 = p_L$ (the projections). Let $G$ and $H$ be the homotopies on $C(M, N)$ induced by $g$ and $h$ given by Lemma 1.2. Define $D_t : C(M, N) \rightarrow C(M, N)$ by

\[
D_t = \begin{cases} 
H_{4t} & \text{if } t \in [0, 1/4], \\
G_{4t-1} & \text{if } t \in [1/4, 1/2], \\
G_{4t-2} & \text{if } t \in [1/2, 3/4], \\
H_{4t-3} & \text{if } t \in [3/4, 1], 
\end{cases}
\]

where $G_t^* = G_{1-t}$ and $H_t^* = H_{1-t}$. Then

\[
D : f \simeq p_Lf \simeq p_Lf p_K \simeq p_L((p_Lf[K] \times \text{id}_{E^\infty}) \simeq (p_Lf[K]) \times \text{id}_{E^\infty}, w3
\]

and

\[
D_1(C(M, N)) = \{ q \times \text{id}_{E^\infty} \mid q \in C(K, L) \} \cong C(K, L)
\]

(by Lemma 1.1) which is a metrizable space [Ku, Theorem 1, p. 93].

Observe that $D(\text{Homo}(M, N) \times_c I) \subset \text{Homo}(M, N)$. So, $D$ deforms $\text{Homo}(M, N)$ in itself into the metrizable subspace $B \equiv \text{Homo}(M, N) \cap \{ q \times \text{id}_{E^\infty} \mid q \in C(K, L) \}$. Recall that since $B$ is metrizable, it also is a subspace of $k\text{Homo}(M, N)$. Now, we would like to deform $B$ into $\text{Homeo}(M, N)$. Consider the associate $g$ of the inclusion $i_B : B \rightarrow \text{Homo}(M, N)$, i.e.,

\[
g : B \times K \times E^\infty \rightarrow B \times L \times E^\infty
\]

defined by $g(b, x, y) = (b, q(x), y)$, where $b = q \times \text{id}_{E^\infty}$. Then, $g$ is continuous [Gr, 8.17], and $g$ is an f.p. homotopy equivalence [Ja, Proposition 7.58]. Therefore, by [L3, Lemma 3.3], $g$ is f.p. homotopic to an f.p. homeomorphism, say $\tilde{g}$. This homotopy induces a homotopy $R : B \times I \rightarrow \text{Homo}(M, N)$ with $R_0 = i_B$ and $R_1(B) \subset \text{Homeo}(M, N)$. Combining $D$ and $R$, we will obtain a desired deformation with a final map $r = R_1D_1$.

Finally, let $i : \text{Homeo}(M) \rightarrow \text{Homo}(M)$ be the natural inclusion. Then we have $ir = r \simeq \text{id}_{\text{Homo}(M)}$. By [L3, Lemma 3.3], $i$ is a weak homotopy equivalence (i.e., $i$
induces isomorphisms on homotopy groups. It follows easily that \( r \) is also a weak homotopy equivalence. \( \Box \)

To conclude this section, we prove the following proposition which will be used in the next section. In fact, the proposition indicates that \( \text{Homeo}(M) \) is not a closed subset of \( \text{Homo}(M) \). Therefore, it is interesting to know whether the deformation in the above theorem can be chosen such that \( \text{Homeo}(M) \) is deformed in itself.

**PROPOSITION 1.4.** Let \( A \) be a closed subset (\( G_\delta \)-subset, resp.) of \( B \in C\mathcal{W}(M) \) (\( B \in C\mathcal{W}(C) \), resp.), and \( (X,p,B) \) an \( E^\infty \)-manifold fiber bundle. If \( f: X \to X \) is an f.p. homotopy equivalence with \( f|X_A \) a continuous injection, then there is an f.p. map \( h: X \to X \) such that \( h|X_{B-A}: X_{B-A} \to X_{B-A} \) is an f.p. homeomorphism and that \( f \) is f.p. homotopic to \( h \) (rel \( X_A \)).

**PROOF.** Similar to the proof of \([L_3, \text{Theorem 3.4}]\), we will assume that \( X = B \times M \), where \( M \) is an \( E^\infty \)-manifold. Since \( A \) is a closed \( G_\delta \)-set in \( B \), we can write \( B - A = \bigcup \{ C_n \mid n = 1, 2, \ldots \} \), where each \( C_n \) is a closed \( G_\delta \)-set in \( B \) and \( C_n \subset \text{(interior of } C_{n+1}) \). Write \( M = \text{dirlim} M_n \) and assume \( C_0 = \emptyset \) and \( M_0 = \emptyset \). We now define by induction a sequence of f.p. homotopies \( H^n: B \times M \times I \to B \times M \) and \( h_n = \lim H^n \) such that

1. \( h_{n-1} \simeq h_n \) (f.p.) rel \((A \times M) \cup (C_{n-1} \times M) \cup (B \times M_{n-1})\),
2. \( h_n|C_1 \times M: C_1 \times M \to C_1 \times M \) is an f.p. homeomorphism, and
3. \( h_n[B \times M_n]: B \times M_n \to B \times M \) is an f.p. embedding.

First, since \( f|A \times M \) is an f.p. embedding, there is from \([L_3, \text{Lemma 3.3}]\) an f.p. homotopy \( F: B \times M \times I \to B \times M \) (rel \( A \times M_1 \)) such that \( F_0 = f \) and \( F_1 \) is an f.p. homeomorphism. Let \( s: B \to I \) be a map with \( s^{-1}(0) = A \) and \( s^{-1}(1) = C_1 \). Define \( H^1: B \times M \times I \to B \times M \) by \( H^1(b, x, t) = F(b, x, s(b)t) \). Then we have

1. \( H^1: (\equiv h_0) \simeq h_1 \) (f.p.) rel \( A \times M \),
2. \( h_1|C_1 \times M = F_1|C_1 \times M \) is an f.p. homeomorphism, and
3. \( h_1[B \times M_1]: B \times M_1 \to B \times M \) is an f.p. embedding, by use of \([L_3, \text{Lemma 1.2}]\).

Second, assume that \( H^{n-1} \) has been defined such that (i)\(_{n-1} \), (ii)\(_{n-1} \), and (iii)\(_{n-1} \) are satisfied. By \([L_3, \text{Lemma 3.3}]\), there is an f.p. homotopy \( G: B \times M \times I \to B \times M \) (rel \( A \times M_n \cup C_{n-1} \times M \cup B \times M_{n-1} \)) such that \( G_0 = h_{n-1} \) and \( G_1 \) is an f.p. homeomorphism. Similar to above, by use of an Urysohn function and \([L_3, \text{Lemma 1.2}]\), we can obtain a homotopy \( H^n \) such that (i)\(_n \), (ii)\(_n \), and (iii)\(_n \) are satisfied.

Finally, define \( h = \lim h_n \); then, \( h \) is well defined and continuous by use of (iii)\(_n \); and, \( h|(B - A) \times M \) is an f.p. homeomorphism by use of (ii)\(_n \) and the condition \( C_n \subset \text{(interior of } C_{n+1}) \). Moreover, a desired homotopy \( H: B \times M \times I \to B \times M \) from \( f \) to \( h \) can be defined by

\[
H(b, x, t) = \begin{cases} 
H^n(b, x, 2^n(t - 1) + 2) & \text{if } t \in J_n, \\
h(x) & \text{if } t = 1,
\end{cases}
\]

where \( J_n = [1 - (1/2^{n-1}), 1 - (1/2^n)] \). Then, \( H \) is well defined and continuous by (i)\(_n \). \( \Box \)

2. **Absolute neighborhood extensor properties of Homeo(M).** We first prove some lemmas that will be used in the proof of the main result of this section, Theorem 2.3.
LEMMA 2.1. Let $K$ be a compact metric space and $M$ an $E^\infty$-manifold. Then the space $M^K \in CW(M)$; moreover, it is an $ANE(M)$ and an $ANE(C)$. Consequently, $M^K$ is an $ANE(CW(M))$ and an $ANE(CW(C))$, and so is $C(K,M)$.

PROOF. The proof is implicit in that of [Hy, Theorem 8.2]. Because of its simplicity in this case, we will give its outline.

Write $M = \text{dirlim} M_n$, where each $M_n$ is a compact ANR metric space with $M_n \subset M_{n+1}$. Observe that

1. by [Hy, Proposition 2.7], $M^K = \bigcup \{(M_n)^K | n = 1, 2, \ldots \}$,
2. by [Hy, Lemma 8.1.b], $(M_n)^K$ is a closed subspace of $(M_{n+1})^K$ for each $n$, and
3. each $(M_n)^K$ is an ANR complete metric space (so, it is an $ANE(C)$ by [M, Theorem 3.1(b)]).

Therefore, by [Hy, Theorem 11.3], $\text{dirlim}(M_n)^K \in CW(M)$ (or an $M$-space of [Hy]) and is an $ANE(M)$. On the other hand, let $P$ be a compact subset of $M^K$ and $e: M^K \times K \to M$ the evaluation map. Then $e(P \times K)$ is a compact subset of $M$. So, there is an $n$ such that $e(P \times K) \subset M_n$; in other words, $P \subset (M_n)^K$ for some $n$. Therefore, $M^K = \text{dirlim}(M_n)^K$ by [Hy, Lemma 5.5]. Also, observe that $M^K$ is an $ANE(C)$ by use of (3).

Now, it follows from [Hy, Theorem 10.2] and its proof that $M^K = \text{dirlim}(M_n)^K$ is an $ANE(CW(M))$ and an $ANE(CW(C))$. Finally, it follows easily that $C(K,M)$ is an $ANE(CW(M))$ and an $ANE(CW(C))$ by use of [Gr, 8.8] and $CW(M) \cup CW(C) \subset CG$ [L3, §0]. \(\square\)

LEMMA 2.2. Let $K$ be a finite complex and $N = K \times E^\infty$. Let $A$ be a closed subset (Gδ-subset, resp.) of $B \in CW(M)$ ($B \in CW(C)$, resp.). Then each map $f: A \to \text{Homeo}(N) \subset \text{Homo}(N)$ has a continuous extension $g: U \to \text{Homo}(N)$ over a neighborhood $U$ of $A$ in $B$; i.e., the associate $g: U \times N \to U \times N$ of $g$ is an f.p. homotopy equivalence.

PROOF. Identify $K$ with $K_0 \subset N$ and let $\tilde{f}: A \times N \to A \times N$ be the associate of $f$. We will first show that $\tilde{f}$ has an f.p. extension $q: V \times N \to V \times N$, where $V$ is a closed neighborhood of $A$ in $B$. Then, we will choose a suitable neighborhood $U$ of $A$ in $V$ such that $\tilde{g} = q|U \times N$ is an f.p. homotopy equivalence.

Define $f_K: A \overset{\tilde{f}}{\to} \text{Homeo}(N) \subset C(N,N) \overset{i_K}{\to} C(K,N)$, where $i_K$ is induced from $i_K$. We have $f_K$ being continuous. It follows from Lemma 2.1 that there is a closed neighborhood $V$ of $A$ in $B$ and an extension $h: V \to C(K,N)$ of $f_K$. Let $h: V \times K \to V \times N$ be its associate [Gr, 8.17] and $R: N \times I \to N$ a strong deformation retraction of $N$ onto $K$. Define $d: V \times N \to V \times N$ by $d = h(id_V \times R_1)$. Then,

$$d|A \times N = h(id_A \times R_1) = \tilde{f}_K(id_A \times R_1) = \tilde{f}(id_A \times R_1) \simeq \tilde{f}(id_A \times id_N) \quad (\text{f.p.}) \text{ since } R_1 \simeq id_N = \tilde{f}.$$ 

Therefore, by [L3, Lemma 1.1], $d$ is f.p. homotopic to a map $q: V \times N \to V \times N$ such that $q|A \times N = \tilde{f}$.
Fix an $a \in A$. Consider $\hat{G} = q(id_N \times f^{-1}_a)$ with its continuous associate $G: V \to C(N, N)$. We have $G(a) = f^{-1}_a = id_N$. From the proof of [L4, Theorem 1], there is an open neighborhood $W$ of $id_N$ in $C(N, N)$ such that for every map $s: X \to W$, its associate $\hat{s}: X \times N \to X \times N$ is f.p. homotopic to $id_{X \times N}$. Let $U_a = G^{-1}(W)$. Then, the restriction $\hat{G}|_{U_a \times N}$ is an f.p. homotopy equivalence; hence, so is $q|_{U_a \times N} = G(id_{U_a} \times f_a)$. Define $U = \bigcup\{U_a \mid a \in A\}$. Then, $\hat{g} = q|U \times N$ is a local f.p. homotopy equivalence; hence, it is an f.p. homotopy equivalence by [Ja, Theorem 5.57], and its associate $g: U \to \text{Homo}(N)$ is an extension of $f$ as desired. □

**THEOREM 2.3.** If $M$ is an $E^\infty$-manifold having homotopy type of a finite complex, then $\text{Homeo}(M)$ is an ANE($\text{CW}(M)$) and an ANE($\text{CW}(C), G_\delta$); hence, so is $k\text{Homeo}(M)$.

**PROOF.** By [He, Theorem C], we can write $M = K \times E^\infty$, where $K$ is a finite complex. Let $A$ be a closed subset ($G_\delta$-subset, resp.) of $B \in \text{CW}(M)$ ($B \in \text{CW}(C)$, resp.), and $f: A \to \text{Homeo}(M)$ a map. Recall that since $A, B \in \text{CW}$, we can use either topology on $\text{Homeo}(M)$ and $\text{Homo}(M)$. From Lemma 2.2, $f$ has an extension $g: U \to \text{Homo}(M)$, where $U$ is a closed neighborhood of $A$ in $B$, such that its associate $\hat{g}: U \times M \to U \times M$ is an f.p. homotopy equivalence. Then, it follows from Proposition 1.4 that there is an f.p. map $h: U \times M \to U \times M$ whose associate $h: U \to C(M, M)$ satisfies the following properties:

(a) $h(U) \subset \text{Homeo}(M)$, and

(b) $h|A = f$.

In other words, $h$ is an extension of $f$ over $U$ into $\text{Homeo}(M)$. □

**REFERENCES**


