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A NOTE ON HAUSDORFF MEASURES OF QUASI-SELF-SIMILAR SETS

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ABSTRACT. Sullivan has demonstrated that quasi-self-similarity provides a useful point of view for the study of expanding dynamical systems. In [**4**, p. 57] he posed the question: Is the Hausdorff measure of a quasi-self-similar set positive and finite in its Hausdorff dimension? This paper answers both parts of this question. In §1 the positivity is established for compact sets, and a lower bound is given for their Hausdorff measure. However, in §2 the finiteness is disproved. In fact, a quasi-self-similar set is constructed for which the Hausdorff measure is actually σ-infinite.

0. Definitions. Let X be a metric space with metric d. A set $S \subset X$ is called k-quasi-self-similar if every small piece of S can be uniformly expanded to a standard size and then mapped k-quasi-isometrically back into S. More precisely:

DEFINITION. A nonvoid set $S \subset X$ is called *k-quasi-self-similar* if there is an $r_0 > 0$ such that, given any ball B with radius $r < r_0$, there exists a map E_B of $B \cap S$ into S such that

$$\frac{1}{k}\frac{r_0}{r}d(x,y) \leq d(E_B(x), E_B(y)) \leq k\frac{r_0}{r}d(x,y)$$

for all $x, y \in B \cap S$. The number r_0 will be called a *standard size* of S.

Hausdorff dimension and (spherical) Hausdorff measure are defined as follows (see [**2**] for a complete discussion).

DEFINITION. Let $r(B)$ denote the radius of the ball B. Then the *δ-Hausdorff measure* of S is

$$H_\delta(S) = \lim_{\varepsilon \to 0}\left[\inf\left\{\sum_{i=1}^{\infty} r(B_i)^\delta : \bigcup_{i=1}^{\infty} B_i \supset S \text{ and } r(B_i) < \varepsilon\right\}\right].$$

Notice that if S is compact one need only be concerned with finite coverings of S. The *Hausdorff dimension* of S is

$$\dim(S) = \inf\{\delta : H_\delta(S) < \infty\}.$$

1. Positivity. The key result of this paper is

LEMMA. *Let $S \neq \varnothing$ be a compact subset of a metric space X and let $\delta = \dim(S)$. Suppose there exist $k > 1$ and $r_0 > 0$ such that for any ball B with radius $r < r_0$, there is a map E_B of $B \cap S$ into S which obeys*

$$\frac{1}{k}\frac{r_0}{r}d(x,y) \leq d(E_B(x), E_B(y))$$

for all $x, y \in B \cap S$. Then $(r_0/2k)^\delta \leq H_\delta(S)$.

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An obvious consequence of this lemma is the promised theorem,

THEOREM 1. *If S is a compact k-quasi-self-similar set with standard size r_0 and $\delta = \dim(S)$, then*

$$(r_0/2k)^\delta \le H_\delta(S).$$

PROOF OF LEMMA. Suppose $H_\delta(S) < (r_0/2k)^\delta$. Then there is a cover of S by a finite number of balls $B_1, \ldots, B_N$ with radii $r_1, \ldots, r_N$ such that

$$r_i < r_0/4k \quad \text{for } i = 1, \ldots, N \tag{1}$$

and

$$\sum_{i=1}^N r_i^\delta < (r_0/2k)^\delta.$$

Choose $\eta < \delta$ such that

$$\sum_{i=1}^N r_i^\eta < (r_0/2k)^\eta. \tag{2}$$

It will be shown that $H_\eta(S) < \infty$ which implies $\eta \ge \dim(S) = \delta$, a contradiction.

For each B_i, there is a map E_i of $B_i \cap S$ into S such that

$$(r_0/kr_i)d(x,y) \le d(E_i(x), E_i(y)) \tag{3}$$

for all $x, y \in B_i \cap S$. Since $E_i[B_i \cap S] \subset S$, $E_i[B_i \cap S]$ is covered by $B_1, \ldots, B_N$. Therefore

$$S \subset \bigcup_{i=1}^N B_i \cap S \subset \bigcup_{i=1}^N \bigcup_{j=1}^N E_i^{-1}[B_j]. \tag{4}$$

By (3) if $x, y \in E_i^{-1}[B_j]$, then

$$\begin{aligned} d(x,y) &\le (kr_i/r_0)d(E_i(x), E_i(y)) \\ &\le (kr_i/r_0)\operatorname{diam}(B_j) = (kr_i/r_0)(2r_j), \end{aligned}$$

so each $E_i^{-1}[B_j]$ is contained in a ball B_{ij} with radius $r_{ij} \le (kr_i/r_0)(2r_j)$. This and (4) imply that we have a new cover of S by the B_{ij}'s with

$$\begin{aligned} \sum_{i,j=1}^N r_{ij}^\eta &\le \sum_{i=1}^N \sum_{j=1}^N (kr_i/r_0)^\eta (2r_j)^\eta \\ &= (2k/r_0)^\eta \sum_{i=1}^N r_i^\eta \sum_{j=1}^N r_j^\eta \\ &< (2k/r_0)^\eta (r_0/2k)^\eta (r_0/2k)^\eta \quad \text{by (2)} \\ &= (r_0/2k)^\eta. \end{aligned}$$

But

$$\begin{aligned} r_{ij} &\le (kr_i/r_0)(2r_j) < (2k/r_0)(r_0/4k)^2 \quad \text{by (1)} \\ &= \tfrac{1}{2}(r_0/4k). \end{aligned}$$

Thus, the new cover has balls of half the size but the same bound on the sum of the ηth powers of the radii.

If this process is iterated a cover can be produced of balls with arbitrarily small radii $\tilde{r}_1, \ldots, \tilde{r}_M$; yet

$$\sum_{i=1}^{M} \tilde{r}_i^{\eta} < (r_0/2k)^{\eta}.$$

Hence $H_\eta(S) \leq (r_0/2k)^\eta < \infty$ and this provides the desired contradiction.

2. A finiteness counterexample. Let T be the interval $[0,1]$ with ends identified to form a circle. For any increasing sequence of positive integers $\{m_n\}$ define the set

$$S(\{m_n\}) = \left\{ \sum_{n=0}^{\infty} a_n 2^{-m_n} : a_n = 0 \text{ or } 1, \ n = 1, 2, \ldots \right\};$$

that is, $S(\{m_n\})$ is the set of numbers whose binary expansions have zeros in all but possibly the m_nth decimal places. In particular, define $S_0 = S(\{2^n\})$; this is the set of numbers with binary expansions of the form

$$.a_0 a_1 0 a_2 000 a_3 0000000 a_4 0 \ldots; \qquad a_i = 0 \text{ or } 1.$$

Now define $E\colon T \to T$ by $E(x) = 2x \pmod 1$ and let $S_n = E^n[S_0]$ (where E^n denotes the n-fold composition of E). For example, S_2 is the set of numbers with binary expansions of the form

$$.0 a_2 000 a_3 0000000 a_4 0 \ldots; \qquad a_i = 0 \text{ or } 1.$$

Finally let $S = \bigcup_{n=0}^{\infty} S_n$. This set provides the anticipated counterexample.

THEOREM 2. (A) *S is compact.*
(B) *S is 2-quasi-self-similar with standard size $\frac{1}{2}$.*
(C) $\dim(S) = 0$.
(D) *H_0 is σ-infinite on S.*

PROOF. (A) Suppose $\{x_n\}$ is a sequence in S with $\lim_{n\to\infty} x_n = x_0$. If

$$\{x_n\} \subset \bigcup_{i=1}^{N} S_i, \qquad N < \infty,$$

then

$$x_0 \in \bigcup_{i=1}^{N} S_i \subset S$$

because each S_i is closed. On the other hand, suppose $\{x_n\}$ cannot be confined to a finite collection of the S_i's. This means that the number of zeros between the first two nonzero entries in the binary expansion of x_n must go to ∞ with n; that is, $\{x_n\}$ converges to a number which has at most one nonzero entry. Hence $x_0 \in \{0, 2^{-1}, 2^{-2}, 2^{-3}, \ldots\}$. In either case $x_0 \in S$, so S is closed and hence compact.

(B) Let (a,b) be an interval in T having length r with $2^{-n-1} \leq r < 2^{-n}$. Then, with $k = 2$ and $r_0 = \frac{1}{2}$,

$$\frac{1}{k}\frac{r_0}{r} < 2^n < k\frac{r_0}{r}.$$

Now suppose $x, y \in S \cap (a,b)$. Then since $d(E^n(x), E^n(y)) = 2^n d(x,y)$ it follows that

$$\frac{1}{k}\frac{r_0}{r} d(x,y) \leq d(E^n(x), E^n(y)) \leq k\frac{r_0}{r} d(x,y).$$

(C) It is an easy exercise to show that

$$\dim S(\{m_n\}) \leq \liminf_{n\to\infty}(m_{n+1} - m_n)^{-1};$$

in particular, since $S_j = S(\{2^n - j\})$,

$$\dim S_j \leq \liminf_{n\to\infty} 2^{-n} = 0.$$

Therefore $\dim S = 0$ because it is a countable union of zero-dimensional sets.

(D) H_0 is just the counting measure, and S is clearly uncountable. Hence H_0 is σ-infinite on S.

NOTES. (1) Counterexamples similar to the set S described above can be constructed in any dimension. For instance, a cheap one-dimensional counterexample is just $S \times [0,1]$.

(2) In the lemma and Theorem 1 of §1 the hypothesis that S is compact cannot be removed. For example, let T and E be as defined in §2, and let Z_0 be any subset of T with $\dim(Z_0) = \delta$ and $H_\delta(Z_0) = 0$. Define

$$Z = \bigcup_{n=0}^{\infty} E^n[Z_0].$$

Then the same proof used for Theorem 2(B) shows that Z is 2-quasi-self-similar with standard size $\frac{1}{2}$. But $\dim(Z) = \delta$ and by countable subadditivity of H_δ, $H_\delta(Z) = 0$.

(3) [**1** and **3**] discuss (strict) self-similarity. In particular, they prove that under certain conditions the Hausdorff measure of a self-similar set is positive and finite in its Hausdorff dimension.

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