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THE K-THEORY OF TRIANGULAR MATRIX RINGS. II

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ABSTRACT. Let T be the upper triangular matrix ring defined by a pair of rings R and S and an R-S-bimodule M. We use the "QP" definition of algebraic K-theory to give a quick proof that the homomorphism

$$\pi_m: K_m(T) \to K_m(R) \oplus K_m(S), \qquad m \geqslant 0,$$

induced by the obvious ring epimorphism is an isomorphism.

Given rings R and S and an R-S-bimodule M, one can construct an upper triangular matrix ring

$$T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix},$$

with the evident addition and multiplication.

Let $\pi: T \to R \oplus S$ be the natural ring epimorphism and let

$$\pi_m: K_m(T) \to K_m(R) \oplus K_m(S), \qquad m \in \mathbf{Z},$$

be the induced homomorphism in K-theory.

It is well known that π_0 and π_1 are isomorphisms and Dennis and Geller [**2**] showed that π_2 is also an isomorphism. In a recent paper [**1**], Jon Berrick and myself confirmed the conjecture that all the π_m are in fact isomorphisms. The arguments in both papers were based on the "BGL^+" definition of K-theory, requiring calculations in the homology of the general linear group.

The purpose of this note is to present an alternative proof of the conjecture, using the "QP" definition of K-theory and the calculus of functors on the category $\mathbf{P}(T)$ of finitely generated projective left T-modules. A similar argument was used in [**3**], but there I relied on homological conditions on the bimodule M which were natural for the rings under investigation.

We consider only the case $m \geqslant 0$.

Let ρ be π followed by projection onto the first factor of $R \oplus S$ and write

$$A = \begin{bmatrix} 0 & M \\ 0 & S \end{bmatrix},$$

so that there is an exact sequence of rings and ideals

$$0 \to A \to T \xrightarrow{\rho} R \to 0.$$

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Regard this sequence first as a sequence of left T-modules. The T-module structure induced on R via ρ is the same as that arising through the column $\left[\begin{smallmatrix} R \\ 0 \end{smallmatrix}\right]$; thus the sequence is split and both R and A are projective left T-modules. We can therefore define endofunctors a and b of $\mathbf{P}(T)$ by $a = A \otimes_T -$ and $b = R \otimes_T -$.

The following equivalences are easily verified:

$$a^2 \simeq a, \quad b^2 \simeq b, \quad ab \simeq 0 \simeq ba.$$

The functor $a \oplus b$ is not the identity functor on $\mathbf{P}(T)$, since it has the wrong effect on homomophisms, but if we think of the exact sequence above as a sequence of right modules, we see that there is an exact sequence

$$0 \to a \to \mathrm{Id} \to b \to 0$$

of exact functors on $\mathbf{P}(T)$.

Write α_m and β_m for the endomorphisms of $K_m(T)$ induced by the functors a and b respectively. A fundamental result of Quillen [**4**, §3, Corollary 3] shows that

$$1 = \alpha_m + \beta_m$$

on $K_m(T)$. Thus α_m and β_m are mutually orthogonal idempotent endomorphisms of $K_m(T)$.

To see that the corresponding direct decomposition of $K_m(T)$ gives the promised isomorphism, we introduce the functors

$$\begin{aligned} r &= R \otimes_T - : \mathbf{P}(T) \to \mathbf{P}(R) \quad (\text{induced by } \rho), \\ w &= \text{incl}: \mathbf{P}(R) \to \mathbf{P}(T), \\ s &= S \otimes_T - : \mathbf{P}(T) \to \mathbf{P}(S) \quad (\text{induced by projection to } S), \end{aligned}$$

and

$$x = A \otimes_S - : \mathbf{P}(S) \to \mathbf{P}(T).$$

There are natural equivalences

$$wr \simeq b, \quad rw \simeq \mathrm{Id}_{\mathbf{P}(R)}, \quad xs \simeq a \quad \text{and} \quad sx \simeq \mathrm{Id}_{\mathbf{P}(S)},$$

corresponding to which there are identities on K-theory:

$$\omega_m \rho_m = \beta_m, \quad \rho_m \omega_m = 1 \text{ on } K_m(R)$$

and

$$\xi_m \sigma_m = \alpha_m, \quad \sigma_m \xi_m = 1 \text{ on } K_m(S).$$

Since $\rho_m \oplus \sigma_m = \pi_m$, the result is proved.

References

1. A. J. Berrick and M. E. Keating, *The K-theory of triangular matrix rings*, Applications of Algebraic K-theory to Algebraic Geometry and Number Theory, Contemp. Math., vol. 55, part I, Amer. Math. Soc., Providence, R. I., 1986, pp. 69–74.

2. R. K. Dennis and S. C. Geller, *K_i of upper triangular matrix rings*, Proc. Amer. Math. Soc. **56** (1976), 73–78.

3. M. E. Keating, *The K-theory of triangular matrix rings and orders*, Algebraic K-theory, Number Theory, Geometry and Analysis, Bielefeld 1982, Lecture Notes in Math., vol. 1046, Springer, Berlin, and New York, 1984, pp. 178–192.

4. D. Quillen, *Higher algebraic K-theory* I, Algebraic K-theory 1, Batelle Institute Conference 1972, Lecture Notes in Math., vol. 341, Springer, Berlin, and New York, 1973, pp. 85–147.

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