DAUGAVET’S EQUATION AND OPERATORS ON $L^1(\mu)$

JAMES R. HOLUB

ABSTRACT. Generalizing a result of Babenko and Pichugov, it is shown that if $T$ is a weakly compact operator on $L^1(\mu)$, where $\mu$ is a $\sigma$-finite nonatomic measure, then $\|I + T\| = 1 + \|T\|$. A characterization of all operators $T$ on $L^1(\mu)$ having this property is also given.

In [2] Daugavet proved that if $T$ is a compact operator on $C(\mu)$, then $\|I + T\| = 1 + \|T\|$, while Babenko and Pichugov [1] subsequently showed the same is true for a compact operator on $L^1(\mu)$ (see [6] for recent extensions). In general, if $X$ is a Banach space and $T \in \mathcal{L}(X)$, then $T$ is said to satisfy Daugavet’s equation [4] if $\|I - T\| = 1 + \|T\|$. This property of an operator arises naturally in the consideration of problems of best approximation in function spaces where it has been utilized by a number of authors (e.g. [4], and the references cited in [1]).

In a recent paper [5] it was shown that Daugavet’s equation actually holds for any weakly compact operator on $C(\mu)$. The purpose of this note is to give an analogous extension of the theorem of Babenko and Pichugov to weakly compact operators by proving that if $\mu$ is a $\sigma$-finite nonatomic measure on a space $S$, and $T$ is a weakly compact operator on $L^1(\mu)$, then $\|I + T\| = 1 + \|T\|$ (Theorem 1). Related results concerning extensions and generalizations of this theorem, including a characterization of all operators $T$ on $L^1(\mu)$ for which $\|I + T\| = 1 + \|T\|$ (Theorem 2) are also given.

We begin with a simple result concerning the norming of operators on $L^1(\mu)$. As usual we denote by $\mathcal{L}(X)$ the space of all bounded linear operators on a Banach space $X$, by $\chi_A$ the characteristic function of a measurable set $A \subset S$, and by $\|f\|_1$ and $\|g\|_\infty$, the norms of functions $f \in L^1(\mu)$ and $g \in L^\infty(\mu)$, respectively. The complement of a set $A$ in $S$ is denoted by $A'$.

**PROPOSITION 1.** Let $T \in \mathcal{L}(L^1(\mu))$. Then given any $\varepsilon > 0$ there is a measurable set $A \subset S$ for which $0 < m(A) < \varepsilon$ and $\|T(\chi_A/m(A))\|_1 > \|T\| - \varepsilon$.

**PROOF.** Since $\mu$ is $\sigma$-finite, $(L^1(\mu))^* = L^\infty(\mu)$. Therefore if $T \in \mathcal{L}(L^1(\mu))$, then $T^* \in \mathcal{L}(L^\infty(\mu))$ and $\|T\| = \|T^*\| = \sup_{\|g\|_\infty = 1} \|T^*g\|_\infty$. Hence given any $\varepsilon > 0$ there is a function $g \in L^\infty(\mu)$ for which $\|g\|_\infty = 1$ and $\|T^*g\|_\infty > \|T\| - \varepsilon/2$. Since $\|T^*g\|_\infty = \text{ess sup}_t |(T^*g)(t)|$ it follows that there is a measurable set $A \subset S$ for which $m(A) > 0$ and $|(T^*g)(t)| > \|T\| - \varepsilon$ for all $t \in S$. Replacing $A$ (if necessary) by a subset $B \subset A$ for which $0 < m(B) < \varepsilon$ and on which the sign of $(T^*g)(t)$ is constant (which may be done since $\mu$ is nonatomic), and replacing $g$ by $(-g)$ if...
this sign is negative, we may assume $0 < m(A) < \varepsilon$ and $(T^*g)(t) > \|T\| - \varepsilon$ for all $t \in A$. It follows that

$$\left\| T \left( \frac{\chi_A}{m(A)} \right) \right\|_1 \geq \left\langle g, T \left( \frac{\chi_A}{m(A)} \right) \right\rangle = \left\langle T^*g, \frac{\chi_A}{m(A)} \right\rangle = \int_S (T^*g)(t) \frac{\chi_A(t)}{m(A)} \, dt$$

$$= \frac{1}{m(A)} \int_A (T^*g)(t) \, dt > \|T\| - \varepsilon,$$

and the proposition is proved.

Using this result we can now obtain the generalization of the theorem of Babenko and Pichugov mentioned earlier.

**THEOREM 1.** If $T$ is a weakly compact operator on $L^1(\mu)$, then $\|I + T\| = 1 + \|T\|.$

**PROOF.** Suppose $T$ is a weakly compact operator on $L^1(\mu)$. Since $\|I + T\| \leq 1 + \|T\|$, we need only prove that $\|I + T\| \geq 1 + \|T\|.$

Let $\varepsilon > 0$ be given. Since $T$ is weakly compact the set $\{Tf \mid \|f\|_1 \leq 1\}$ is weakly sequentially compact in $L^1(\mu)$ by the Eberlein-Smulian theorem (e.g. [3, p. 430]), and hence the set $\{Tf \mid \|f\|_1 \leq 1\}$ is also a weakly sequentially compact subset of $L^1(\mu)$ [3, p. 293]. It follows that for the given $\varepsilon > 0$ there exists a $\delta > 0$ so that if $A$ is any measurable subset of $S$ with $m(A) < \delta$, then $\int_A |(Tf)(t)| \, dt < \varepsilon/4$ for all $f \in L^1(\mu)$ satisfying $\|f\|_1 \leq 1$ [3, p. 294].

By Proposition 1 there is a set $A \subset S$ for which $0 < m(A) < \delta$ and

$$\|T(\chi_A/m(A))\|_1 > \|T\| - \varepsilon/2.$$

Setting $f = \chi_A/m(A)$ we have $\|f\|_1 = 1$ and

$$\|f + T f\|_1 = \int_S |(f + Tf)(t)| \, dt - \int_A |(Tf)(t)| \, dt + \int_{A'} |(Tf)(t)| \, dt$$

$$\geq \int_A |f(t)| \, dt - \int_A |(Tf)(t)| \, dt + \int_{A'} |(Tf)(t)| \, dt$$

$$= \|f\|_1 + \int_S |(Tf)(t)| \, dt - 2 \int_A |(Tf)(t)| \, dt$$

$$\geq 1 + \left( \|T\| - \frac{\varepsilon}{2} \right) - \frac{2\varepsilon}{4} = 1 + \|T\| - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary it follows that $\|I + T\| \geq 1 + \|T\|$, and the theorem is proved.

We have stated and proved Theorem 1 in a straightforward manner (rather than deriving it as a corollary to a more general, if less transparent, result) in order to emphasize its simplicity and its relationship to the results of [1, 2, and 5]. We now show that one can actually characterize those operators $T$ in $L(L^1(\mu))$ for which $\|I + T\| = 1 + \|T\|$ in terms of a condition on $T$ which has an interesting relationship to the norming property of $T$ expressed in Proposition 1. Before stating and proving the characterization we note the following simple (and probably well known) result concerning the norm of an adjoint operator.

**PROPOSITION 2.** Let $X$ be a Banach space and $T \in L(X)$. Then $\|T^*\| = \sup_{f \in E} \|T^*f\|$, where $E$ denotes the set of extreme points of the unit ball of $X^*$.
PROOF. By definition, \( \|T^*\| \geq \sup_{f \in E} \|T^* f\| \). Suppose \( \|T^*\| > \sup_{f \in E} \|T^* f\| + \varepsilon \) for some \( \varepsilon > 0 \). Choose \( f_0 \in X^* \) and \( x_0 \in X \) so that \( \|f_0\| = \|x_0\| = 1 \) and \( \langle T^* f_0, x_0 \rangle > \|T^*\| - \varepsilon/4 \). By the Krein-Milman theorem [3, p. 440] there is a convex combination \( \sum_{i=1}^{n} \lambda_i f_i \) of members of \( E \) for which \( \langle f_0 - \sum_{i=1}^{n} \lambda_i f_i, T x_0 \rangle < \varepsilon/4 \). It follows that

\[
\left| \langle T^* f_0, x_0 \rangle \right| < \frac{\varepsilon}{4} + \sum_{i=1}^{n} \lambda_i \left| \langle T^* f_i, x_0 \rangle \right|
\leq \frac{\varepsilon}{4} \sup_{1 \leq i \leq n} \left| \langle T^* f_i, x_0 \rangle \right| \leq \frac{\varepsilon}{4} + \sup_{f \in E} \|T^* f\|,
\]

so

\[
\|T^*\| - \frac{\varepsilon}{4} < \left| \langle T^* f_0, x_0 \rangle \right| < \frac{\varepsilon}{4} + \sup_{f \in E} \|T^* f\| < \frac{\varepsilon}{4} + (\|T^*\| - \varepsilon).
\]

That is, \( \|T^*\| < \|T^*\| - \varepsilon/2 \), a contradiction. Therefore \( \|T^*\| \leq \sup_{f \in E} \|T^* f\| \), and the proposition is proved.

Now in Proposition 1 we showed that if \( T \in \mathcal{L}(L^1(\mu)) \), then for any \( \varepsilon > 0 \) there is a set \( B \subset S \) for which \( 0 < m(B) < \varepsilon \) and

\[
\int_S \left| T \left( \frac{\chi_B}{m(B)} \right) t \right| dt > \|T\| - \varepsilon.
\]

Thus there is some set \( A \subset S \) (e.g. the set \( A = \{ t \mid T(\chi_B/m(B))(t) > 0 \} \)) for which

\[
\int_A \left( \frac{\chi_B}{m(B)} \right) (t) dt - \int_A T \left( \frac{\chi_B}{m(B)} \right) t dt > \|T\| - \varepsilon,
\]

although in general the sets \( A \) and \( B \) are unrelated. We now show that the operators \( T \in \mathcal{L}(L^1(\mu)) \) for which \( \|I + T\| = 1 + \|T\| \) are precisely those for which the sets \( A \) and \( B \) in the above inequality can always be assumed to satisfy the relation \( B \subset A \).

**THEOREM 2.** An operator \( T \in \mathcal{L}(L^1(\mu)) \) has the property that \( \|I + T\| = 1 + \|T\| \) \iff for every \( \varepsilon > 0 \) there are measurable sets \( A \) and \( B \) in \( S \) for which \( B \subset A \) and

\[
\int_A \left( \frac{\chi_B}{m(B)} \right) (t) dt - \int_A T \left( \frac{\chi_B}{m(B)} \right) t dt > \|T\| - \varepsilon.
\]

**PROOF.** Suppose \( T \in \mathcal{L}(L^1(\mu)) \) and \( \|I + T\| = 1 + \|T\| \). Then \( T^* \in \mathcal{L}(L^\infty(\mu)) \) and \( 1 + \|T^*\| = \|I + T\| = 1 + \|T\| + \|I + T^*\| \), so by Proposition 2 we have \( 1 + \|T^*\| = \sup_{g \in E} \|g + T^* g\|_\infty \), where \( E \) is the set of extreme points of the unit ball in \( L^\infty(\mu) \). But \( g \in E \Rightarrow \|g(t)\| = 1 \) a.e., so for any \( \varepsilon > 0 \) there is a corresponding function \( g_0 \in L^\infty(\mu) \) for which \( \|g_0(t)\| = 1 \) a.e. and \( \|g_0 + T^* g_0\|_\infty > 1 + \|T^*\| - \varepsilon/2 \). Since

\[
\|g_0 + T^* g_0\|_\infty = \text{ess sup}_t |g_0(t) + (T^* g_0)(t)|
\]

there must be a set \( B \subset S \) of positive measure for which

\[
\inf_{t \in B} |g_0(t) + (T^* g_0)(t)| > 1 + \|T^*\| - \varepsilon.
\]

It follows, since \( |g_0(t)| = 1 \) a.e. and \( \|T^* g_0\|_\infty \leq \|T^*\| \), that \( (T^* g_0)(t) \) must have the same sign as \( g_0(t) \) and \( |(T^* g_0)(t)| > \|T^*\| - \varepsilon \) for all \( t \in B \). Hence we may assume \( g_0(t) = 1 \) and \( (T^* g_0)(t) > \|T^*\| - \varepsilon \) for all \( t \in B \), since there is a subset of
B with positive measure on which either \( g_0 \) or \((-g_0)\) has this property. If we set \( f_0 = \chi_B/m(B) \), then
\[
\int_S g_0(t)(Tf_0)(t)\,dt = \langle g_0, Tf_0 \rangle = \langle T^*g_0, f_0 \rangle = \int_S (T^*g_0)(t)f_0(t)\,dt
\]
\[
= \frac{1}{m(B)} \int_B (T^*g_0)(t)\,dt > \|T^*\| - \varepsilon
\]
(since \((T^*g_0)(t) > \|T^*\| - \varepsilon\) for \( t \in B \)). Setting \( A = \{ t | g_0(t) = 1 \} \) we see that \( B \subseteq A \) and also that
\[
\int_A T \left( \frac{\chi_B}{m(B)} \right)(t)\,dt - \int_{A'} T \left( \frac{\chi_B}{m(B)} \right)(t)\,dt
\]
\[
= \int_S T \left( \frac{\chi_B}{m(B)} \right)(t)g_0(t)\,dt = \int_0^1 (Tf_0)(t)g_0(t)\,dt > \|T^*\| - \varepsilon.
\]
Since \( \|T\| = \|T^*\| \) it follows that \( T \) satisfies the stated condition.

Conversely, suppose \( T \) satisfies the condition and \( \varepsilon > 0 \) is given. Then there are measurable sets \( A \) and \( B \) for which \( B \subseteq A \) and
\[
\int_A T \left( \frac{\chi_B}{m(B)} \right)(t)\,dt - \int_{A'} T \left( \frac{\chi_B}{m(B)} \right)(t)\,dt > \|T\| - \varepsilon.
\]
Again, for simplicity of notation, let \( f_0 = \chi_B/m(B) \). If we define
\[
g_0(t) = \begin{cases} 1 & \text{if } t \in A, \\ -1 & \text{if } t \notin A,
\end{cases}
\]
then it follows that
\[
\langle Tf_0, g_0 \rangle = \int_S (Tf_0)(t)g_0(t)\,dt > \|T\| - \varepsilon,
\]
where \( g_0(t) = 1 \) for all \( t \in B \) since \( B \subseteq A \). Therefore
\[
\|f_0 + Tf_0\|_1 \geq \langle f_0 + Tf_0, g_0 \rangle = \langle f_0, g_0 \rangle + \langle Tf_0, g_0 \rangle
\]
\[
\geq \int_S f_0(t)g_0(t)\,dt + (\|T\| - \varepsilon) = 1 + \|T\| - \varepsilon,
\]
since \( f_0 = \chi_B/m(B) \) and \( g_0(t) = 1 \) for all \( t \in B \). It follows that \( \|I + T\| \geq 1 + \|T\| \), and since \( \|I + T\| \leq 1 + \|T\| \) we see that \( \|I + T\| = 1 + \|T\| \). The theorem is completely proved.

Although Theorem 2 is generally rather awkward to apply, it is convenient to use in certain particular cases. As an example we consider the \( L^1(\mu) \)-analogue of a result of Franchetti and Cheney [4] concerning Daugavet’s equation in the space \( C(S) \).

Let \( k \) be a measurable subset of \( S \) for which \( 0 < m(k) < m(S) \) and let \( R: L^1(\mu) \rightarrow L^1[k] \) be the restriction map. Suppose \( E: L^1[k] \rightarrow L^1(\mu) \) is some bounded linear extension mapping. Then \( ER \) is a projection on \( L^1(\mu) \) with \( \|ER\| = \|E\| \).

In the setting of the result of Franchetti and Cheney mentioned above \( k \) is taken instead to be a closed, nonopen, subset of \( S \), \( R \) and \( E \) are taken to be the analogous restriction and extension operators on \( C(S) \) and \( C(k) \), respectively, and it is shown that for \( \text{any} \) such \( E \) the projection \( ER \) on \( C(S) \) satisfies Daugavet’s equation—i.e.
\[ \| I - ER \| = 1 + \| E \|. \] In contrast, for the \( L^1(\mu) \) case we have

**Proposition 3.** There is no bounded linear extension mapping \( E: L^1[k] \to L^1(\mu) \) for which the projection \( ER \) on \( L^1(\mu) \) satisfies Daugavet’s equation.

**Proof.** Let \( K \subseteq S \) for which \( 0 < m(k) < m(S) \), let \( R: L^1(\mu) \to L^1[k] \) be the restriction map, and let \( E: L^1[k] \to L^1(\mu) \) be any bounded linear extension mapping. If \( \| I - ER \| = 1 + \| E \| \), then by Theorem 2 it must be true that given any \( \varepsilon > 0 \) there exist measurable subsets \( A \) and \( B \) of \( S \) with \( B \subseteq A \) for which

\[
\left| \int_A (-ER) \left( \frac{\chi_B}{m(B)} \right) \, dt - \int_A' (-ER) \left( \frac{\chi_B}{m(B)} \right) \, dt \right| > \| E \| - \varepsilon.
\]

That is, since \( R(\chi_B) = \chi_{B \cap k} \),

\[
\frac{\| E \| m(B \cap k)}{m(B)} \geq \int_{A'} E \left( \frac{\chi_{B \cap k}}{m(B)} \right) \, dt - \int_A E \left( \frac{\chi_{B \cap k}}{m(B)} \right) \, dt > \| E \| - \varepsilon.
\]

In particular, \( m(B \cap k)/m(B) \geq 1 - \varepsilon \), since \( \| E \| \geq 1 \). Moreover, since \( B \subseteq A \) we then have

\[
\int_{A'} E \left( \frac{\chi_{B \cap k}}{m(B)} \right) \, dt - \int_{A \setminus B} E \left( \frac{\chi_{B \cap k}}{m(B)} \right) \, dt - \int_{B \setminus k} E \left( \frac{\chi_{B \cap k}}{m(B)} \right) \, dt > \| E \| - \varepsilon,
\]

or

\[
\left[ \int_{A'} E \left( \frac{\chi_{B \cap k}}{m(B)} \right) \, dt - \int_{A \setminus B} E \left( \frac{\chi_{B \cap k}}{m(B)} \right) \, dt - \int_{B \setminus k} E \left( \frac{\chi_{B \cap k}}{m(B)} \right) \, dt \right] - \frac{m(B \cap k)}{m(B)} \int_{B \cap k} E \left( \frac{\chi_{B \cap k}}{m(B \cap k)} \right) \, dt > \| E \| - \varepsilon.
\]

Now \( A', A \setminus B \), and \( B \setminus k \) are disjoint, \( \| \chi_{B \cap k}/m(B) \|_1 \leq 1 \), and \( m(B \cap k)/m(B) \geq 1 - \varepsilon \), so since \( E(\chi_{B \cap k})(t) = \chi_{B \cap k}(t) \) for \( t \in B \cap k \) it follows from this that \( \| E \| - (1 - \varepsilon) > \| E \| - \varepsilon \), or \( 1 < 2\varepsilon \). Since \( \varepsilon > 0 \) arbitrarily we have reached a contradiction, and the proposition is established.

**Remarks.** (1) It was observed in [5] that if \( S \) is a compact Hausdorff space and \( T \in L(C(S)) \), then either \( \| I + T \| \) or \( \| I - T \| = 1 + \| T \| \). In particular, if \( \mu \) is a positive measure and \( T \in L(L^\infty(\mu)) \), then (since \( L^\infty(\mu) \) is isometric to some such \( C(S) \) [3, p. 445]) either \( \| I + T \| \) or \( \| I - T \| = 1 + \| T \| \). Consequently, if \( T \in L(L^1(\mu)) \), then \( T^* \in L(L^\infty(\mu)) \), so either \( \| I + T^* \| \) or \( \| I - T^* \| = 1 + \| T^* \| = 1 + \| T \| \). It follows, then, that either \( \| I + T \| \) or \( \| I - T \| = 1 + \| T \| \) for any \( T \in L(L^1(\mu)) \).

(2) The condition that \( \mu \) be nonatomic is necessary for the validity of our results. For example, if \( T = (-1)e_1 \otimes e_1 \in L(l^1) \), then \( T \) is compact; yet

\[
\| I + T \| = \sup_n \| e_n - (e_1, e_n)e_1 \| = 1 \neq 1 + \| T \|.
\]

In the same way, if the measure space \( (S, \Sigma, \mu) \) has an atom \( A \in \Sigma \), then by a similar construction there is a one-dimensional operator \( T \in L(\mu) \) for which \( \| I + T \| \neq 1 + \| T \| \).

**References**


DEPARTMENT OF MATHEMATICS, VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY, BLACKSBURG, VIRGINIA 24061