THE COMPLEMENT OF THE STABLE MANIFOLD
FOR ONE DIMENSIONAL ENDOMORPHISMS

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ABSTRACT. Let \( N \) denote either the circle \( S^1 \) or the closed interval \( I = [0,1] \) and let \( f \) be a \( C^1 \) endomorphism of \( N \). Let \( \Sigma(f) \) be the complement of the union of the stable manifolds of the sinks of \( f \). In this paper we give necessary and sufficient conditions for \( \Sigma(f) \) to consist of eventually periodic points.

1. Introduction. Let \( N \) denote either the circle \( S^1 \) or the closed interval \( I = [0,1] \) and let \( f : N \rightarrow N \) be a \( C^1 \) differentiable map (endomorphism). As usual we say that \( x \in N \) is a periodic point of \( f \) if \( f^n(x) = x \) for some \( n \geq 1 \). In this case we say that \( x \) is hyperbolic if \( |(f^n)'(x)| \neq 1 \), a sink if \( |(f^n)'(x)| < 1 \), and a source if \( |(f^n)'(x)| > 1 \). Let \( P(f) \) and \( P_c(f) \) denote respectively the set of periodic points and the set of sinks. The set \( P(f) \setminus P_c(f) \) will be denoted by \( P_e(f) \).

Let \( x \in P(f) \). The stable set of \( x \), \( W^s(x) \), is defined as the set of points \( y \) such that \( \lim_{n \rightarrow +\infty} d(f^n(x), f^n(y)) = 0 \). Each stable set consists of countably many disjoint (possibly degenerate) intervals. The stable manifold of \( f \), \( \Delta(f) \), is defined by \( \Delta(f) = \bigcup W^s(x) \), where the union is taken over all the sinks of \( f \). We let \( \Sigma(f) = N \setminus \Delta(f) \). Note that \( \Delta(f) \) and \( \Sigma(f) \) are invariant under \( f \).

In studying the dynamics of an endomorphism \( f \) of \( N \), the sets \( P(f) \) and \( \Sigma(f) \) play an important role. In general, \( \bigcup_{n=0}^\infty f^{-n}(P_c(f)) \subseteq \Sigma(f) \), but equality need not hold. In this paper we give necessary and sufficient conditions such that \( \Sigma(f) = \bigcup_{n=0}^\infty f^{-n}(P_c(f)) \).

**THEOREM.** Let \( f \) be a \( C^1 \) endomorphism of \( N \). \( \Sigma(f) = \bigcup_{n=0}^\infty f^{-n}(P_c(f)) \) if and only if \( P(f) \) is closed and nonempty and for every \( x \in P_e(f) \), \( W^s(x) = \bigcup_{n=0}^\infty f^{-n}(\{x\}) \).

**COROLLARY.** Let \( f : N \rightarrow N \) be a \( C^1 \) endomorphism with all the periodic points hyperbolic. \( \Sigma(f) = \bigcup_{n=0}^\infty f^{-n}(P_c(f)) \) if and only if \( P(f) \) is finite and nonempty.

We remark that the theorem is similar in flavor to results in the same setting which give necessary and sufficient conditions for the nonwandering set to equal the periodic points set \([3, 5, 7, 8]\) and for the nonwandering set to equal the chain recurrent set \([2, 4]\). These results will be the fundamental tools for the proof of the theorem.

2. Proof of the Theorem. We begin by recalling some basic concepts and establishing preliminary results.

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Let $f: N \to N$ be a $C^1$ endomorphism of $N$. For any point $x \in N$, we let $\mathcal{O}_f(x) = \bigcup_{n=-\infty}^{\infty} f^n\{x\}$ and $\mathcal{O}_f^+(x) = \bigcup_{n=0}^{\infty} f^n(x)$. $x$ is said to be eventually periodic if $\mathcal{O}_f^+(x)$ is finite or equivalently if some iterate of $x$ is periodic.

The $\omega$-limit of a point $x \in N$ is defined by $\omega(x) = \{y \in N; y \in \overline{\mathcal{O}_f^+(x)}\}$. $x$ is said to be recurrent if $\omega(x) = \mathcal{O}_f^+(x)$. A point $x$ is called nonwandering if for any neighborhood $U$ of $x$ there is an integer $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. Let $R(f)$ and $\Omega(f)$ denote the sets of recurrent and nonwandering points respectively.

Let $p$ be a periodic point of (least) period $n$ which is not a critical point of $f^n$. Let $k = n$ if $f^n$ preserves orientation at $p$, and $k = 2n$ if $f^n$ reverses orientation at $p$. We call $p$ an expanding periodic point if there is an open neighborhood $V_p$ of $p$ such that for each $x \in V_p$, $|f^k(x) - p| > |x - p|$.

We say that $f$ has a horseshoe if for some $n$ there are disjoint closed intervals $J$ and $K$ such that each $J$ and $K$ $f^n$-covers both $J$ and $K$. When $f$ has a horseshoe as above, the $f^n$-invariant set

$$H = \bigcap_{i=0}^{\infty} f^{-in}(J \cup K)$$

has the full one-sided shift on two symbols as a continuous factor $[1, 3]$.

**Lemma 2.1.** Let $f$ be an endomorphism of $N$. If $\Sigma(f) = \bigcup_n f^{-n}(\mathcal{P}_c(f))$, then $f$ has no horseshoes.

**Proof.** By contradiction, suppose that $f$ has a horseshoe. Then, for some $n$ there is a subset $H$ of $N$, such that $f^n(H) = H$ and there is a topological semiconjugacy $h$ of $f^n: H \to H$ onto the full (one-sided) shift $\sigma$ on two symbols.

We claim that $R(\sigma) = \mathcal{P}(\sigma)$. In fact, let $x \in R(\sigma)$. Choose $y \in H$ such that $h(y) = x$. If $y \in \Delta(f^n/H)$, then there is a sink $z$ of $f^n/H$ such that $\omega(y) = \mathcal{O}_{f^n}^+(z)$ and so $\omega(x) = \mathcal{O}_{f}^+(h(z))$. This implies that $x \in \mathcal{P}(\sigma)$, because $h(z)$ is a periodic point of $\sigma$. If $y \in \Sigma(f^n/H)$, then, by hypothesis, $y$ is an eventuall periodic point of $f^n/H$ and so $x$ is an eventually periodic point of $\sigma$. This implies that $x \in \mathcal{P}(\sigma)$. Therefore $R(\sigma) = \mathcal{P}(\sigma)$. But this is a contradiction because $\sigma$ is topologically transitive. So $f$ has no horseshoes.

**Lemma 2.2.** Let $f$ be a $C^1$ endomorphism of $N$. Let $y \in \mathcal{P}_c(f) \setminus \partial N$ such that $W^s(y) = \mathcal{O}_f(y)$ and $y \in \omega(x)$ for some $x \neq y$. If $y$ is not an expanding periodic point of $f^2$ then there exists an open interval $(a, b)$ containing $y$ and such that either $(a, y) \cap \mathcal{O}_{f^2}^+(x) = \emptyset$ and $f^2$ is expanding on $(y, b)$, or $(y, b) \cap \mathcal{O}_{f^2}^+(x) = \emptyset$ and $f^2$ is expanding on $(a, y)$.

**Proof.** Without loss of generality we may assume that $y$ is a fixed point of $f$. By hypothesis, there exists an interval $(a_1, b_1)$ containing $y$ and such that $g = f^2$ is strictly monotone increasing on $(a_1, b_1)$.

We will assume that the iterates of $x$ have a subsequence $(f^{m_i}(x))$ in $(a_1, y)$ such that $(f^{m_i}(x))$ is monotonically increasing to $y$ (the proof is similar if the subsequence is contained in $(y, b_1)$).

We claim that there exists a sequence $(z_n)$ in $(a, y)$ such that $z_n$ is a fixed point of $g$ and $(z_n)$ approaches $y$. Hence, there exists $n_k$ such that
\[z_n < f^{n_k}(x) < z_m \text{ for some } n \text{ and } m. \] Hence \(z_n < g^l(f^{n_k}(x)) < z_m\) for every \(l\). This implies that \(\lim_{n \to \infty} f^{n_k}(x) \neq y\). This is a contradiction and proves that for some interval \((a, y)\), \(g\) is expanding on \((a, y)\). Since \(y\) is not an expanding point of \(g\), there exists an interval \((y, b)\) such that \(g\) is not expanding on \((y, b)\). Thus by the argument above, there exists an interval \((y, b)\) with \(b \leq b\) and such that \((y, b) \cap \mathcal{O}^+_g(x) = \emptyset\). This ends the proof of the lemma.

Using the same arguments as Lemma 2.2, one can easily prove the following.

**Lemma 2.3.** Let \(y \in P(e) \cap \partial I\) such that \(W^s(y) = \mathcal{O}_f(y)\) and \(y \in \omega(x)\) for some \(x \neq y\). Then \(y\) is an expanding periodic point of \(f\).

Now we are ready to prove the theorem. First suppose \(\Sigma(f) = \bigcup_{n=0}^{\infty} f^{-n}(P(e))\). Since either \(\Sigma(f)\) or \(\Delta(f)\) is nonempty, so is \(P(f)\).

We claim that \(\Omega(f) = P(f)\). Otherwise, by Lemma 2.1 [1, Theorem A; 3, Proposition 2; 7, Lemma 3], any \(x \in \Omega(f) \setminus P(f)\) has an infinite further orbit. This is a contradiction, because \(x \in \Sigma(f)\) and, by hypothesis, \(\Sigma(f)\) consists of eventually periodic points. Hence \(\Omega(f) = P(f)\) and this implies that \(P(f)\) is closed.

Let \(x \in P(e)\). We will show that \(W^s(x) = \mathcal{O}_f(x)\). Let \(y \in W^s(x)\). Since \(x \notin \Delta(f)\), \(y \in \Sigma(f)\), and consequently \(y\) is an eventually periodic point of \(f\). This implies that \(y \in \mathcal{O}_f(x)\). Therefore \(W^s(x) = \mathcal{O}_f(x)\).

Now suppose that \(P(f)\) is closed and nonempty and \(W^s(x) = \mathcal{O}_f(x)\) for every \(x \in P(e)\). We shall show that if \(y \in \Sigma(f)\), then \(y\) is an eventually periodic point of \(f\). The proof adapts a technique from [4, Theorem B].

By contradiction, suppose that \(y\) has an infinite further orbit. By [3, 6, 7, 8], \(\Omega(f) = P(f)\) and consequently \(\omega(y) \subset P(e)\). Let \(y_1 \in \omega(y)\) and let \(n_1 = 2j_1\), where \(j_1\) is the period of \(y_1\). We choose a neighborhood \(I_1\) of \(y_1\) as follows.

If \(y_1\) is an expanding periodic point of \(f^2\), we choose \(I_1 = (y_1 - \varepsilon_1, y_1 + \varepsilon_1)\) for some \(\varepsilon_1 > 0\) such that \(f^{m_1}\) is expanding on \(I_1 \subset N\).

If \(y_1\) is not an expanding periodic point of \(f^2\), we have, by Lemma 2.3, that \(y_1 \notin \partial N\). By Lemma 2.2, we may assume without loss of generality that there is an interval \(I_1 = (y_1 - \varepsilon_1, y_1 + \varepsilon_1)\) such that \((y_1 - \varepsilon_1, y_1 + \varepsilon_1) \cap \mathcal{O}^+_{f^2}(x) = \emptyset\) and \(f^{m_1}\) is expanding on \((y_1, y_1 + \varepsilon_1)\).

Let \(W_1\) and \(V_1\) be neighborhoods of \(y_1\) such that \(W_1 \subset V_1 \subset I_1\), \(f^{m_1}(W_1) \subset V_1\), and the diameter \(l(V_1)\) of \(V_1\) is less than \(\varepsilon_1/3\). We claim that there is a point \(y_2\) in \(\omega(y) \cap (\overline{V_1} \setminus \overline{W_1})\). If \(y_1\) is an expanding point, the claim is obvious. If \(y_1\) is not an expanding point, then by the assumption above, there is \(\tilde{y}_2 \in \omega(y)\) such that \(\tilde{y}_2 \in W_1 \cap (y_1, y_1 + \varepsilon_1)\). Since \(f^{m_1}\) is expanding on \((y_1, y_1 + \varepsilon_1)\), \(f^{k_{m_1}}(y_2) \in \overline{V_1} \setminus \overline{W_1}\) for some positive integer \(k\). This proves the claim.

Let \(O_1\) be the union of neighborhoods of diameter \(l(W_1)\) about each periodic point \(z \notin W_1\) whose period is at most \(n_1\) and let \(K_1 = N \setminus (O_1 \cup W_1)\). Then \(y_2\) is in the interior of \(K_1\). Let \(n_2 = 2j_2\), where \(j_2\) is the period of \(y_2\) and note that \(n_2 > n_1\). Choose \(I_2, V_2\) and \(W_2\) as above with \(I_2 \subset \overline{W}_2\) contained in the interior of \(K_1\). There is a point \(y_3\) in \(\omega(y) \cap (\overline{V_2} \setminus \overline{W_2})\). Let \(O_2\) be the union of neighborhoods of diameter \(l(W_2)\) about each periodic point \(z \notin W_2\) whose period \(k\) satisfies \(n_2 < k \leq n_2\). Let \(K_2 = N \setminus (O_2 \cup W_1 \cup O_2 \cup W_2)\) and note that \(y_3\) is in the interior of \(K_2\).

Define \(K_n\) inductively as above, and let \(G_n = K_n \cap \omega(y)\). Then \(G_n\) is a decreasing family of nonempty compact sets, so \(\bigcap_{n=1}^{\infty} G_n\) is nonempty. Any point in the intersection is in \(\omega(Y)\) but is not periodic, a contradiction.
Hence \( y \) is an eventually periodic point of \( f \) and this ends the proof of the Theorem.

3. Proof of the Corollary. First suppose that \( P(f) \) is finite and nonempty. It is clear that \( P(f) \) is closed. By hypothesis \( P_e(f) \) consists of sources. This implies that \( W^s(x) = \mathcal{O}_f(x) \) for every \( x \in P_e(f) \). Hence by the Theorem, \( \Sigma(f) = \bigcup_{n=0}^{\infty} f^{-n}(P_e(f)) \).

Now suppose \( \Sigma(f) = \bigcup_{n=0}^{\infty} f^{-n}(P_e(f)) \). By the Theorem, \( P(f) \) is closed and nonempty. This implies that \( P_e(f) \) is closed because it consists of sources. It follows that there exists \( n \in \mathbb{N} \) and \( c > 1 \) such that \( |(f^n)'(x)| > c \) for all \( x \in P_e(f) \). This implies that \( f|_{P_e(f)} : P_e(f) \to P_e(f) \) is an expansive homeomorphism. This, together with the fact that \( P_e(f) \) is totally disconnected, implies by [6, p. 92], that \( f|_{P_e(f)} \) is conjugate to a subshift \( \sigma \) of finite type. It follows that all the points in the phase space of \( \sigma \) are periodic. This implies that \( P(\sigma) \) is finite and so \( P_e(f) \) is finite. It follows that \( P_c(f) \) is also finite and therefore \( P(f) \) is finite.

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