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TWO-RELATOR GROUPS WITH PRESCRIBED COHOMOLOGICAL DIMENSION

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ABSTRACT. For each integer $n \geqslant 2$, an example is constructed of a 2-generator, 2-relator presentation of a group $G(n)$ of cohomological dimension $n+1$.

This note is a response to the following question, which was put to me by J.-P. Serre.

Lyndon's Identity Theorem [3] implies that the cohomological dimension of a one-relator group is either 0, 1, 2, or ∞. Is there a corresponding statement for two-relator groups?

It is very easy to construct two-relator groups of cohomological dimension 0, 1, 2, 3 and ∞. Serre (unpublished) has constructed one of dimension 4, and Brown and Geoghegan [1] have shown that the two-relator group

$$\left\langle x, y \middle| [x, x^y], [x, x^{y^2}] \right\rangle$$

is torsion-free, of type FP_∞, and of cohomological dimension ∞. (Here a^b denotes $b^{-1}ab$ and $[a, b]$ denotes $a^{-1}b^{-1}ab$.)

On the other hand, if there is a generator which does not appear in one relator, but essentially appears in the other, then the presentation is "reducible" in the sense of [2]. If in addition neither relator is a proper power modulo the other, then the group has cohomological dimension at most 2.

In fact, we construct examples of two-relator groups of any given finite cohomological dimension. Let $G(n)$ be the group

$$\langle x, y | [x, x^y], [y^n, x] x^y \rangle.$$

and $G(n)'$ its commutator subgroup. Then $G(n)/G(n)'$ is clearly infinite cyclic. We prove

PROPOSITION. *$G(n)'$ is free abelian of rank n.*

It follows that $G(n)$ is poly-(infinite cyclic) of Hirsch number (and hence cohomological dimension) $n+1$.

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PROOF OF PROPOSITION. Clearly $G(n)'$ is the normal closure in $G(n)$ of x, so is generated by the elements $x_i = y^{-i}xy^i$ ($i \in \mathbf{Z}$). Indeed the second defining relation can be rewritten in terms of these new generators as $x_n = x_0x_1$, so $G(n)'$ is actually generated by $x_0, \ldots, x_{n-1}$, subject to the relations $[x_i, x_{i+1}]$ ($i \in \mathbf{Z}$).

Thus $G(n)'/G(n)''$ is free abelian of rank n, and so it suffices to prove that $G(n)'' = \{1\}$, in other words that $G(n)'$ is abelian. To do this it suffices in turn (using the automorphisms $x_i \to x_{i+1}$ of $G(n)'$) to prove that $[x_0, x_i] = 1$ for all $i > 0$, and we shall prove the latter statement by induction on i. Now $[x_0, x_1] = 1$ by the defining relations, so suppose inductively that $[x_0, x_i] = 1$ for all i with $0 \leqslant i < k$. Then

$$[x_0, x_{k-1}] = [x_1, x_{k-1}] = [x_1, x_k] = 1,$$

so

$$[x_0, x_k] = [x_0x_1, x_{k-1}x_k] = [x_n, x_{n+k-1}] = y^{-n}[x_0, x_{k-1}]y^n = 1$$

as claimed.

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