ISOMORPHISMS OF GRAPH GROUPS

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ABSTRACT. Given a graph $X$, define the presentation $PX$ to have generators the vertices of $X$, and a relation $xy = yx$ for each pair $x, y$ of adjacent vertices. Let $GX$ be the group with presentation $PX$, and given a field $K$, let $KX$ denote the $K$-algebra with presentation $PX$. Given graphs $X$ and $Y$ and a field $K$, it is known that the algebras $KX$ and $KY$ are isomorphic if and only if the graphs $X$ and $Y$ are isomorphic. In this paper, we use this fact to prove that if the groups $GX$ and $GY$ are isomorphic, then so are the graphs $X$ and $Y$.

Given a graph $X$ with vertex set $V(X)$, we define the presentation $PX$ to be that having as generators the elements of $V(X)$, with a defining relation $vw = vw$ for each pair $v$ and $w$ of adjacent vertices of $X$. $PX$ can be regarded as a presentation of a group $GX$, or of a $K$-algebra $KX$ over a field $K$. In [1], Kim, Makar-Limanov, Neggers, and Roush proved that if the algebras $KX$ and $KY$ are isomorphic, then so are the graphs $X$ and $Y$. (In their formulation, two generators commute provided they are not adjacent in the graph. This is sufficient for our purposes, since if two graphs are isomorphic, so are their complements.)

Let $K$ be a field. In this note we will show that if the groups $GX$ and $GY$ are isomorphic, then so are the algebras $KX$ and $KY$, thus demonstrating:

THEOREM. If the groups $GX$ and $GY$ are isomorphic, then so are the graphs $X$ and $Y$.

Let $f: GX \rightarrow GY$ be an isomorphism. Denote by $G_2X$ the quotient group $GX/[GX, (GX')]$. Then $f$ induces an isomorphism $f_2: G_2X \rightarrow G_2Y$. Let $V(X)$ and $V(Y)$ be totally ordered, and denote both orderings by $<$. We will not distinguish by notation between a vertex of $X$, the corresponding element of $GX$, and the image of this element in $G_2X$. For each vertex $x$ of $X$, $f_2(x)$ can be written uniquely in the form $y_1^1y_2^2 \cdots y_n^nC_x$, where $C_x$ is an element of the commutator subgroup $(G_2Y)'$, $y_1 < y_2 < \cdots < y_n$, and the integers $a_r$ are all nonzero. Define $f_*(x) = a_1y_1 + a_2y_2 + \cdots + a_ny_n$. We will show that the function $f_*: X \rightarrow KY$ extends to a homomorphism $f_*: KX \rightarrow KY$ by showing that if $xx' = x'x$ is a relation of $PX$, then $f_*(x)f_*(x') = f_*(x)f_*(x')$ in $KY$.

LEMMA. The commutators $\{[x_i, x_j] \mid x_i < x_j \text{ and } x_i \text{ and } x_j \text{ are not adjacent in } X\}$ of $G_2X$ are linearly independent.

PROOF. Consider the exact sequence

$$1 \rightarrow N \rightarrow FX \rightarrow GX \rightarrow 1$$

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associated with the presentation $PX$. Since $N$ is a subgroup of $FX'$, there is an exact sequence

$$1 \rightarrow \hat{N} \rightarrow (F_2X)' \rightarrow (G_2X)' \rightarrow 1$$

where $\hat{N}$ denotes the image of $N$ in $F_2X$. By the Basis Theorem, $(F_2X)'$ is a free abelian group with a basis consisting of the commutators $\{[x_i, x_j] \mid x_i < x_j\}$. Since $\hat{N}$ is the subgroup of $(F_2X)'$ generated by the collection $\{[x_i, x_j] \mid x_i < x_j, x_i$ and $x_j$ adjacent\}, the assertion follows.

Suppose that $x$ and $x'$ are adjacent vertices of $X$. Let $f_2(x) = y_1^{a_1}y_2^{a_2}\cdots y_n^{a_n}C_x$ and $f_2(x') = y_1^{b_1}y_2^{b_2}\cdots y_n^{b_n}C_{x'}$, with $y_1 < y_2 < \cdots < y_n$, where we allow some of the exponents $a_r$ and $b_s$ to be 0 so that we may use the same elements of $V(Y)$ in both expressions. Writing $(G_2Y)'$ additively, and recalling that in $G_2Y$, commutators are central, we have

$$0 = [f_2(x), f_2(x')] = \sum_{r,s} [y_r^{a_r}, y_s^{b_s}]$$

$$= \sum_{r,s} a_r b_s [y_r, y_s] = \sum_{r<s} (a_r b_s - a_s b_r) [y_r, y_s].$$

Thus, by the Lemma, if $y_r$ and $y_s$ are nonadjacent vertices of $Y$, then $a_r b_s - a_s b_r = 0$. Clearly, $f_*(x) = \sum_r a_r y_r$ and $f_*(x') = \sum_s b_s y_s$, so that

$$f_*(x)f_*(x') - f_*(x')f_*(x) = \sum_{r,s} (a_r b_s - b_r a_s)y_ry_s.$$

If $y_r$ and $y_s$ are adjacent vertices of $Y$, then $y_r y_s = y_s y_r$ in $KY$, so the net coefficient of $y_r y_s$ in this sum is $(a_r b_s - b_r a_s) + (a_s b_r - b_s a_r) = 0$. If $y_r$ and $y_s$ are nonadjacent vertices of $Y$, then the net coefficient of $y_r y_s$ is $a_r b_s - b_r a_s$, and we saw above that this must be 0 when $y_r$ and $y_s$ are not adjacent. Finally, for each $r$, the coefficient of $y_r^2$ is $a_r b_r - b_r a_r = 0$. Thus, $f_*(x)f_*(x') - f_*(x')f_*(x) = 0$, so $f_*$ extends to a homomorphism from $KX$ to $KY$.

It is now an easy matter to check that for each vertex $x$ of $X$, $(f^{-1})_* f_*(x) = x$, and for each vertex $y$ of $Y$, $f_*(f^{-1})_*(y) = y$, so that $f_*$ is in fact an isomorphism.

The following shorter proof of our theorem has been pointed out by an anonymous reviewer: Define $G_0 = GX$ and $G_{n+1} = [G_n, G]$ for $n \geq 0$, and let $LX$ denote the Lie algebra $K \otimes \sum G_n/G_{n+1}$. Then the algebra $KX$ is the universal enveloping algebra of $LX$, so that if $GX$ and $GY$ are isomorphic groups, then the Lie algebras $LX$ and $LY$, and hence the algebras $KX$ and $KY$, are isomorphic.

REFERENCES