LEVEL SETS FOR FUNCTIONS CONVEX IN ONE DIRECTION

JOHNNY E. BROWN

Abstract. Goodman and Saff conjectured that if \( f \) is convex in the direction of the imaginary axis then so are the functions \( \frac{1}{n} f(rz) \) for all \( 0 < r < \sqrt{2} - 1 \), i.e., the level sets \( f(|z| < r) \) are convex in the direction of the imaginary axis for \( 0 < r < \sqrt{2} - 1 \). A weak form of this conjecture is proved and a question of Brannan is answered negatively.

Let \( U_r = \{ z : |z| < r \} \) and let \( S \) denote the class of all functions \( f(z) = z + a_2 z^2 + \cdots \) analytic and univalent in \( U = U_1 \). For \( f \in S \) there are several geometric properties possessed by \( f(U) \) that are inherited by its level sets \( G_r = f(U_r) \) for all \( 0 < r < 1 \). For example if \( f(U) \) is either convex, starlike, or close-to-convex, then so are its level sets \( G_r \) for all \( 0 < r < 1 \).

An analytic function \( f \) is said to be convex in the direction of a line \( L_\theta : te^{i\theta} \) \((-\infty < t < \infty)\) if the intersection of \( f(U) \) with each line parallel to \( L_\theta \) is either a connected set or empty. Let \( CIA \) denote those functions \( f \) for which \( f(U) \) is convex in the direction of the imaginary axis with \( f(0) = 0 \) and \( f'(0) = 1 \). Since \( CIA \) functions are close-to-convex, they are univalent. It came as a bit of a surprise when Hengartner and Schober \[4\] constructed an example where \( f \in CIA \) but the corresponding level sets \( G_r \) were not convex in the direction of the imaginary axis for all \( r \) sufficiently close to 1. A more quantitative result was obtained by Goodman and Saff \[3\]. They were able to prove that for each \( \sqrt{2} - 1 < r < 1 \) there exists an \( f \in CIA \) for which \( \frac{1}{n} f(rz) \notin CIA \). Hence they conjectured that if \( f \in CIA \) then \( \frac{1}{n} f(rz) \in CIA \) for all \( 0 < r < \sqrt{2} - 1 \), i.e., the level sets \( G_r \) are convex in the direction of the imaginary axis for \( 0 < r < \sqrt{2} - 1 \).

In \[1, \text{Problem 6.53}\] Brannan asked whether or not: If \( f \in CIA \) and \( \frac{1}{n} f(r_0 z) \notin CIA \) for some \( 0 < r_0 < 1 \), does this imply that \( \frac{1}{n} f(rz) \notin CIA \) for all \( r_0 < r < 1 \)? This question was motivated by the example constructed by Hengartner and Schober \[4\]. In this note we prove a weaker form of the Goodman-Saff conjecture and answer Brannan’s question. Our main result is the following theorem.

Theorem. If \( f \in CIA \) then there exists a set \( I \subseteq [0, \pi] \) of positive measure such that \( \frac{1}{n} f(rz) \) is convex in the directions \( L_\theta : te^{i\theta} \) \((-\infty < t < \infty)\) for all \( 0 < r < \sqrt{2} - 1 \) and all \( \theta \in I \).
The proof of this theorem depends on a representation formula for CIA functions and a result relating the total variation of the argument along a curve to the number of times the curve intersects lines through the origin.

**Lemma A (Royster and Ziegler [8]).** A function \( f \) belongs to CIA if and only if

\[
\text{Re} \left\{ \frac{-izf'(z)}{h_s(e^{-in}z)} \right\} > 0, \quad z \in U,
\]

for some \( 0 \leq \mu, \nu \leq \pi \) where \( h_s(z) = z/[1 - (2\cos \nu)z + z^2] \).

A function \( f \) analytic in \( U \) is said to be starlike in the direction \( L_\theta: t e^{i\theta} (\theta \in \mathbb{R}) \) if the intersection of \( f(U) \) with \( L_\theta \) is a single segment, half-line, or \( L_\theta \). The following result gives a relation between these functions and functions convex in one direction. Our formulation follows easily from the results in [7].

**Lemma B (Robertson [7]).** If \( zf'(z) \) is starlike in the direction \( L_\theta: t e^{i\theta} (\theta \in \mathbb{R}) \) for \( |z| < r \), then \( f(z) \) is convex in the direction \( L_{\theta + \pi/2} \) for \( |z| < r \).

Finally we need a counting result given in [9] and given as a problem in [2, p. 215].

**Lemma C.** If \( \phi(z) \) is analytic in \( |z| < r \) and \( 0 \notin \phi(|z| = r) \), then

\[
\int_0^{2\pi} \left| \text{Re} \left\{ \frac{z\phi'(z)}{\phi(z)} \right\} \right| d\theta = \int_0^\pi n(\psi) \, d\psi, \quad |z| = r,
\]

where \( n(\psi) \) is the number of times the line \( te^{i\psi} (\theta \in \mathbb{R}) \) intersects the curve \( \phi(|z| = r) \).

Although this result is geometrically obvious, for completeness sake we give a proof using the Banach indicatrix.

**Proof of Lemma C.** By replacing \( \phi(z) \) by \( \phi(e^{i\alpha}z) \), if necessary, we may assume \( \phi(r) > 0 \). Let \( 0 = \theta_1 < \theta_2 < \cdots < \theta_{n-1} < \theta_n = 2\pi \) be all those angles such that \( \phi(re^{i\theta_k}) > 0 \) and \( \text{Arg} \phi(re^{i\theta}) \) varies continuously from 0 to \( 2\pi \) for \( \theta_k \leq \theta \leq \theta_{k+1} \) for each \( 1 \leq k \leq n - 1 \). Note that if the curve \( \phi(|z| = r) \) winds around the origin more than once, then \( n \geq 3 \). For each interval \( \theta_k \leq \theta \leq \theta_{k+1} \), \( 1 \leq k \leq n - 1 \), let \( H_k(\theta) = \text{Arg} \phi(re^{i\theta}) \). Hence \( 0 \leq H_k(\theta) \leq 2\pi \). Let \( m_k(\psi) \) be equal to the number of roots of the equation \( H_k(\theta) = \psi, 0 \leq \psi \leq 2\pi \). The function \( m_k \) is the Banach indicatrix of the function \( H_k \) and thus

\[
\int_0^{2\pi} m_k(\psi) \, d\psi = \int_{\theta_k}^{\theta_{k+1}} |dH_k(\theta)|, \quad 1 \leq k \leq n - 1
\]

(see Natanson [6, p. 225] for example). Now since \( |dH_k(\theta)| = |\text{Re} \left\{ z\phi'(z)/\phi(z) \right\}|d\theta \), it follows that

\[
\int_0^{2\pi} m(\psi) \, d\psi = \int_0^{2\pi} \left| \text{Re} \left\{ \frac{z\phi'(z)}{\phi(z)} \right\} \right| d\theta,
\]
where \( m(\psi) = m_1(\psi) + m_2(\psi) + \cdots + m_n(\psi) \) is the number of times the curve \( \phi(|z| = r) \) intersects the ray \( te^{i\psi} \) \((0 < t < \infty)\) counting multiplicities. Finally observe that
\[
\int_0^{2\pi} m(\psi) \, d\psi = \int_0^{\pi} m(\psi) + m(\psi + \pi) \, d\psi = \int_0^{\pi} n(\psi) \, d\psi
\]
and so the proof of the lemma is complete.

If \( \phi(z) \) is analytic and vanishes only for \( z = 0 \) in \(|z| < r\) then we assert that \( n(\psi) \) is a positive even integer function except for finitely many \( \psi \). To see this, suppose that \( n(\psi) \) is odd for infinitely many \(|z_k| = r\). Since \( \phi(|z| = r) \) is an analytic Jordan curve surrounding the origin it follows that \( \text{Re}\{z_k\phi(z_k)/\phi(z_k)\} = 0 \). The function \( \omega(z) = z\phi'(z)/\phi(z) \) is analytic in \(|z| < r\) and the curve \( \omega(|z| = r) \) meets the imaginary axis an infinite number of times. Hence we can conclude that \( \omega(z) \equiv i\lambda \) for some \( \lambda \in \mathbb{R} \), but since \( \omega(0) = 1 \) a contradiction is reached. The assertion is proved.

**Proof of Theorem.** Fix \( 0 < r < \sqrt{2} - 1 \). Observe first that
\[
Q(\xi) \equiv \text{Re}\left\{ \frac{\xi h_\nu(\xi)}{h_\nu(\xi)} \right\} = \text{Re}\left\{ \frac{1 - \xi^2}{1 - (2\cos \nu)\xi^2} \right\},
\]
where \( h_\nu \) is given in Lemma A, is harmonic in \( U \), \( Q(0) = 1 \), and \( Q(\xi) > 0 \). If \( p(z) \) is analytic in \( U \), \( p(0) = 1 \), and \( \text{Re} \, p(z) > 0 \), then the following estimate was given by Libera [5]:
\[
|zp'(z)| \leq 2r \frac{2r}{1 - r^2}, \quad |z| = r,
\]
for any \( \beta \in \mathbb{R} \).

Let \( f \in \mathcal{C}^{1,1}_A \). Then by Lemma A we get
\[
(2) \quad zf'(z) = h_\nu(e^{-i\nu z})[\cos \mu + ip(z) \sin \mu]
\]
for some \( 0 \leq \mu, \nu \leq \pi \) and some function \( p \) analytic in \( U \) with \( p(0) = 1 \), \( \text{Re} \, p > 0 \). If \( \phi(z) = zf'(z) \) then from (2) we obtain
\[
(3) \quad \frac{z\phi'(z)}{\phi(z)} = \left\{ \frac{(e^{-i\mu z})h_\nu(e^{i\nu z})}{h_\nu(e^{-i\mu z})} \right\} + \left\{ \frac{zp'(z)}{p(z) - i\cot \mu} \right\}
\]
(if \( \mu = 0 \) or \( \pi \) the last term is not present). It follows from (3) and (1) that
\[
\int_0^{2\pi} \left| \text{Re} \, \frac{z\phi'(z)}{\phi(z)} \right| d\theta \leq \int_0^{2\pi} \left| \text{Re} \, \left\{ \frac{(e^{-i\mu z})h_\nu(e^{i\nu z})}{h_\nu(e^{-i\mu z})} \right\} \right| d\theta + \int_0^{2\pi} \left| \frac{zp'(z)}{p(z) - i\cot \mu} \right| d\theta
\]
\[
\leq 2\pi + 2\pi \left( \frac{2r}{1 - r^2} \right) = 2\pi \left( \frac{1 + 2r - r^2}{1 - r^2} \right),
\]
where \(|z| = r\). By hypothesis, \( r < \sqrt{2} - 1 \) and hence
\[
\int_0^{2\pi} \left| \text{Re} \, \left( \frac{z\phi'(z)}{\phi(z)} \right) \right| d\theta < 4\pi.
\]
Now from Lemma C we get

\[ \int_0^\pi n(\psi) \, d\psi < 4\pi. \]

As pointed out earlier, since \( \phi(z) = zf'(z) \) is analytic in \( |z| < r \) and vanishes only at \( z = 0 \), \( n(\psi) \geq 2 \) and \( n(\psi) \) is a positive even integer function except for a finite set of \( \psi \), say \( E_0 = \{ \psi_1, \psi_2, \ldots, \psi_n \} \). The inequality (4) thus gives \( n(\psi) < 4 \) for all \( \psi \in J \) for some set \( J \subseteq [0, \pi] \) with positive measure. Hence \( n(\psi^*) = 2 \) for all \( \psi^* \in J \setminus E_0 = I^* \) and so the function \( \phi(z) = zf'(z) \) is starlike in the directions \( L_{\psi^*} \) for all \( \psi^* \in I^* \). We now apply Lemma B to conclude that for each such direction \( L_{\psi^*} \), the function \( f(z) \) is convex in the direction \( L_{\psi} \), where \( \psi = \psi^* + \pi/2 \) for \( |z| < r \). This completes the proof of the theorem.

We now turn to the question of Brannan. Specifically we find a function \( F \in CIA \) with \( \frac{1}{r_0} F(r_0 z) \notin CIA \) but \( \frac{1}{r_1} F(r_1 z) \in CIA \) for some \( r_0 < r_1 < 1 \). The function

\[ F(z) = \frac{z - Az^2}{(1 - Bz)^2}, \]

where \( A = e^{2i\alpha} \cos \alpha \) and \( B = e^{i\alpha} \), belongs to \( CIA \) and maps \( U \) onto the exterior of a vertical slit lying along the line \( \text{Re} \, w = -\cos \alpha/2 \) (see [3]). If \( (\partial / \partial \theta) \text{Re} \, F(re^{i\theta}) \) has exactly two sign changes in \( [0, 2\pi) \), then \( \frac{1}{r} F(rz) \in CIA \); while if it has four sign changes then \( \frac{1}{r} F(rz) \notin CIA \). Following [3] we see that by replacing \( z \) by \( ze^{-i\alpha} \), the number of sign changes of \( (\partial / \partial \theta) \text{Re} \, F(re^{i\theta}) \) is the same as the number of sign changes of \( Q(\theta) \) in \( [0, 2\pi) \), where

\[ Q(\theta) = (1 + r^2) \sin(\theta - \alpha) + r [3 \sin \alpha - \sin(2\theta + \alpha)]. \]

In what follows, let \( \alpha = 2.6 \). Suppose first that \( r = r_0 = 0.5 \). Then since \( Q(0) = Q(2\pi) = -0.1288 \ldots, \ Q(1) = 0.0206 \ldots, \ Q(2) = -0.0883 \ldots, \) and \( Q(\pi) = 1.1598 \ldots \) we can conclude that \( \frac{1}{r_0} F(r_0 z) \notin CIA \).

Suppose next that \( r = r_1 = 0.7 \). Table 1 contains \( Q \) and \( Q' \) correct to six and three decimal places, respectively. Note that since \( Q \) and \( Q' \) have at most four zeros in \( [0, 2\pi) \), we see that \( Q \) must have a zero in the interval \( (0, 1) \), and in \( (4.2, 2\pi) \) and possibly in the interval \( (2.05, 2.06) \). It is easy to check that for \( 2.05 < \theta < 2.06 \) we get

\[ Q(\theta) > (1.49)(-0.523) + 0.7[3(0.515) - 0.424] > 0.005. \]

Hence \( Q \) has exactly two zeros in \( [0, 2\pi) \) and so \( \frac{1}{r_1} F(r_1 z) \in CIA \) and \( r_1 > r_0 \).

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( Q(\theta) )</th>
<th>( Q'(\theta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-0.046391</td>
<td>-0.0777</td>
</tr>
<tr>
<td>1</td>
<td>0.288771</td>
<td>0.113</td>
</tr>
<tr>
<td>2.05</td>
<td>0.020353</td>
<td>-0.009</td>
</tr>
<tr>
<td>2.06</td>
<td>0.020351</td>
<td>0.009</td>
</tr>
<tr>
<td>4.2</td>
<td>3.271910</td>
<td>-0.049</td>
</tr>
<tr>
<td>2\pi</td>
<td>-0.046391</td>
<td>-0.0777</td>
</tr>
</tbody>
</table>
Finally, it should be pointed out that our proof of the theorem cannot yield the Goodman-Saff conjecture directly as it is not sensitive to direction. We have proved the weaker conjecture that if $f$ is convex in one direction then $\frac{1}{2}f(rz)$ is convex in one direction for all $0 < r < \sqrt{2} - 1$. If we could rotate CIA functions then the Goodman-Saff conjecture would follow immediately. The classes of convex, starlike, and close-to-convex functions are rotationally invariant, but the class of CIA functions clearly is not. Despite this drawback, we believe the Goodman-Saff conjecture is true.

The author wishes to thank Y. J. Leung and Glenn Schober for a helpful conversation.

REFERENCES


Department of Mathematics, Purdue University, West Lafayette, Indiana 47907