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ARENS PRODUCT AND THE ALGEBRA OF DOUBLE MULTIPLIERS. II

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ABSTRACT. We show that the algebra of double multipliers of a certain Banach algebra A can be embedded in the second conjugate space A^{**} of A. This generalizes the previous work by the author for B^*-algebras.

1. Introduction. Let A be a semisimple Banach algebra and $M(A)$ the algebra of double multipliers on A. The author recently showed that $M(A)$ is isomorphic to $(A^{**}, \circ)$ if and only if A has the following properties: (1) A is Arens regular, (2) A has an approximate identity, and (3) $\pi(A)$ is an ideal of $(A^{**}, \circ)$ [**10**, Theorem, p. 442]. In general, not many Banach algebras possess all these properties. The purpose of this paper is to establish an embedding of $M(A)$ in the second conjugate space A^{**} of A for a semisimple Banach algebra A with an approximate identity. In fact, we show that $M(A)$ is isometrically isomorphic to M^{**}/N^{**}, where $M^{**} = \{F \in A^{**}\colon\ F \circ \pi(x)$ and $\pi(x) \circ F \in \pi(A)$ for all $x \in A\}$ and $N^{**} = \{F \in A^{**}\colon F(f \circ \pi) = F(x \circ' f) = 0$ for all $x \in A$ and $f \in a^*\}$. The main idea of the proof of this result is essentially contained in the proofs of [**8**, Lemma 2.1, p. 80 and **10**, Theorem, p. 422]. This embedding was studied in [**1** and **8**] for B^*-algebras.

2. Notation and preliminaries. Definitions not explicitly given are taken from Rickart's book [**6**].

Let A be a Banach algebra. Then A^* and A^{**} will denote the first and second conjugate spaces of A, and π the canonical map of A into A^{**}. The two Arens products on A^{**} are defined in stages according to the following rules (see [**2**]). Let $x, y \in A$, $f \in A^*$, and $F, G \in A^{**}$.

Define $f \circ x$ by $(f \circ x)(y) = f(xy)$. Then $f \circ x \in A^*$.

Define $G \circ f$ by $(G \circ f)(x) = G(f \circ x)$. Then $G \circ f \in A^*$.

Define $F \circ G$ by $(F \circ G)(f) = F(G \circ f)$. Then $F \circ G \in A^{**}$.

Define $x \circ' f$ by $(x \circ' f)(y) = f(yx)$. Then $x \circ' f \in A^*$.

Define $f \circ' F$ by $(f \circ' F)(x) = F(x \circ' f)$. Then $f \circ' F \in A^*$.

Define $F \circ' G$ by $(F \circ' G)(f) = G(f \circ' F)$. Then $F \circ' G \in A^{**}$.

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A^{**} is a Banach algebra under the products $F \circ G$ and $F \circ' G$ and π is an algebra isomorphism of A into $(A^{**}, \circ)$ and $(A^{**}, \circ')$. In general, $\circ$ and $\circ'$ are distinct on A^{**}. If they agree on A^{**}, then A is called Arens regular.

The following properties of the Arens products will be used often in the rest of this paper.

LEMMA 2.1. *Let A be a Banach algebra. Then for all $x \in A$, $f \in A^*$, and $F, G \in A^{**}$, we have*

(1) $\pi(x) \circ F = \pi(x) \circ' F$ *and* $F \circ \pi(x) = F \circ' \pi(x)$.

(2) *If* $\{F_\beta\} \subset A^{**}$ *and* $F_\beta \to F$ *weakly in* A^{**}, *then* $F_\beta \circ G \to F \circ G$ *and* $G \circ' F_\beta \to G \circ' F$ *weakly.*

(3) $F(x \circ' f) = (F \circ \pi(x))(f)$.

PROOF. (1) is proved in [**2**, p. 843] and (2) in [**2**, p. 842].

(3) It is easy to see that $\pi(x) \circ f = x \circ' f$. Therefore

$$(F \circ \pi(x))(f) = F(\pi(x) \circ f) = F(x \circ' f).$$

This completes the proof of the lemma.

We say that a Banach algebra A has an approximate identity if there exists a net $\{e_\alpha\}$ in A such that $\|e_\alpha\| \leqslant 1$ for all α and $x = \lim e_\alpha x = \lim x e_\alpha$ for all $x \in A$.

Let A be a semisimple Banach algebra. A pair (T_1, T_2) of operators from A to A is called a double multiplier (centralizer) on A provided that $x(T_1 y) = (T_2 x) y$ for all $x, y \in A$. It is knowns that T_1 and T_2 are continuous linear operators on A such that $T_1(xy) = (T_1 x) y$ and $T_2(xy) = x(T_2 y)$. The set $M(A)$ of all double multipliers on A is a Banach algebra with identity and A can be identified as a two-sided ideal of $M(A)$ (see [**4** and **5**]).

In this paper, all algebras and spaces under consideration are over the complex field.

3. Extensions of double multipliers. Let A be a semisimple Banach algebra. We show that each double multipler $T = (T_1, T_2)$ on A can be extended to a double multiplier on $(A^{**}, \circ)$. In fact, for all $x \in A$ and $f \in a^*$, we define

$$(f \circ T_1)(x) = f(T_1 x) \quad \text{and} \quad (f \circ T_2)(x) = f(T_2 x).$$

Then $f \circ T_1$ and $f \circ T_2 \in A^*$. For all $F \in A^{**}$, we define

$$T_1^{\#}(F)(f) = F(f \circ T_1) \quad \text{and} \quad T_2^{\#}(F)(f) = F(f \circ T_2).$$

Then $T_1^{\#}(f)$ and $T_2^{\#}(F) \in A^{**}$.

THEOREM 3.1. *Let A be a semisimple Banach algebra. Then for each $T = (T_1, T_2) \in M(A)$, $T^{\#} = (T_1^{\#}, T_2^{\#})$ is a continuous double multiplier on $(A^{**}, \circ)$ and $T^{\#} \mid \pi(A) = T$.*

PROOF. Let F and $G \in A^{**}$. Then by Goldstine's Theorem, there exist nets $\{x_m\}$ and $\{y_n\}$ in A such that $\pi(x_m) \to F$ and $\pi(y_n) \to G$ weakly in A^{**}. Then for all $f \in A^*$, we have

$$\begin{aligned}(F \circ T_1^{\#}(G))(f) &= \lim_m (T_1^{\#}(G) \circ f)(x_m) = \lim_m G((f \circ x_m) \circ T_1) \\ &= \lim_m \lim_n ((f \circ x_m) \circ T_1)(y_n) \\ &= \lim_m \lim_n f((T_2 x_m) y_n) = \lim_m G(f \circ T_2 x_m) \\ &= \lim_m ((G \circ f) \circ T_2)(x_m) = (T_2^{\#}(F) \circ G)(f).\end{aligned}$$

Therefore $F \circ T_1^{\#}(G) = T_2^{\#}(F) \circ G$. Similarly, we can show that $T_1^{\#}(F \circ G) = T_1^{\#}(F) \circ G$ and $T_2^{\#}(F \circ G) = F \circ T_2^{\#}(G)$. It is easy to see that $\|T_1^{\#}\| \leqslant \|T_1\|$ and $\|T_2^{\#}\| \leqslant \|T_2\|$. Therefore $\|T^{\#}\| \leqslant \|T\|$. Since $T_1^{\#}(\pi(x)) = \pi(T_1 x)$ and $T_2^{\#}(\pi(x)) = \pi(T_2 x)$, we have $T^{\#} \mid \pi(A) = T$ and the theorem is proved.

4. The subalgebras M^{**} and N^{**} of $(A^{**}, \circ)$. For any Banach algebra A, set

$$M^{**} = \{F \in A^{**}\colon F \circ \pi(x) \text{ and } \pi(x) \circ F \in \pi(A) \text{ for all } x \in A\}$$

and

$$N^{**} = \{F \in A^{**}\colon F(f \circ x) = F(x \circ' f) = 0 \text{ for all } x \in A \text{ and } f \in A^*\}.$$

Then $\pi(A)$ is a two-sided ideal of $(A^{**}, \circ)$ if and only if $M^{**} = A^{**}$.

LEMMA 4.1. *Let A be a Banach algebra. Then M^{**} and N^{**} are closed subalgebras of $(A^{**}, \circ)$ and $(A^{**}, \circ')$. Also N^{**} is a two-sided ideal in M^{**} and $N^{**} = \{F \in A^{**}\colon A^{**} \circ F = F \circ' A^{**} = (0)\}$.*

PROOF. For any $F \in N^{**}$, easy calculations show that $\pi(A) \circ F = (0)$ and $F \circ \pi(A) = (0)$. Hence $A^{**} \circ F = F \circ' A^{**} = (0)$ and so $N^{**} = \{F \in A^{**}\colon A^{**} \circ F = F \circ' A^{**} = (0)\}$. Thus if $F, G \in N^{**}$, then $\pi(A) \circ (F \circ G) = (F \circ G) \circ \pi(A) = (0)$ and so $F \circ G \in N^{**}$. Similarly, $F \circ' G \in N^{**}$. It is easy to check that M^{**} and N^{**} are closed subalgebras in $(A^{**}, \circ)$ and $(A^{**}, \circ')$ and N^{**} is a two-sided ideal of M^{**}. Therefore the lemma is proved.

In the rest of this section, A will be a semisimple Banach algebra with an approximate identity $\{e_\alpha\}$. It follows from [**3**, Proposition 7, p. 146] that $(A^{**}, \circ)$ has a right identity I. Also I is a left identity for $(A^{**}, \circ')$.

THEOREM 4.2. *Let A be a semisimple Banach algebra with an approximate identity $\{e_\alpha\}$. Then the Arens products $\circ$ and $\circ'$ agree on M^{**}/N^{**} and M^{**}/N^{**} is a Banach algebra with an identity.*

PROOF. Let $F, G \in M^{**}$, $f \in A^*$, and $x \in A$. Then by Cohen's Factorization Theorem [**3**, Theorem 10, p. 61], $x = yz$ with $y, z \in A$. Hence

$$\begin{aligned}(F \circ G - F \circ' G)(f \circ x) &= (F \circ G - F \circ' G)((f \circ y) \circ z) \\ &= (\pi(z) \circ F \circ G - \pi(z) \circ F \circ G)(f \circ y) = 0.\end{aligned}$$

Similarly, $(F \circ G - F \circ' G)(x \circ' f) = 0$. Therefore $F \circ G - F \circ' G \in N^{**}$ and so $\circ$ and $\circ'$ agree on M^{**}/N^{**}. Since I is a right identity for $(A^{**}, \circ)$ and a left identity for $(A^{**}, \circ)$ and $I \in M^{**}$, it follows that M^{**}/N^{**} is a Banach algebra with an identity.

REMARK. For a certain Banach algebra A, N^{**} is the radical of $(A^{**}, \circ)$ (for example, see [**9**, Theorem 4.1, p. 440]).

The following result is essentially contained in the proof of [**8**, Lemma 2.1, p. 80]. It is useful in the next section.

LEMMA 4.3. *Let A be a semisimple Banach algebra with an approximate identity $\{e_\alpha\}$, $T = (T_1, T_2) \in M(A)$, and $x \in A$. Then*

(1) $T_1^{\#}(I) \circ \pi(x) = T_2^{\#}(I) \circ \pi(x) = \pi(T_1 x)$.

(2) $\pi(x) \circ T_1^{\#}(I) = \pi(x) \circ T_2^{\#}(I) = \pi(T_2 x)$.

(3) $T_1^{\#}(I) = T_2^{\#}(I) + N$ *for some* $N \in N^{**}$.

(4) *If* $T_1^{\#}(I)$ *and* $T_2^{\#}(I)$ *are both in* N^{**}, *then* $T = 0$.

PROOF. (1) For all $f \in A^*$, we have

$$\begin{aligned}\big(T_1^{\#}(I) \circ \pi(x)\big)(f) &= I\big((\pi(x) \circ f) \circ T_1\big) = \lim_\alpha \big((\pi(x) \circ f) \circ T_1\big)(e_\alpha) \\ &= \lim_\alpha f(T_1(e_\alpha x)) = \lim_\alpha (f \circ T_1)(e_\alpha x) \\ &= (f \circ T_1)(x) = f(T_1 x).\end{aligned}$$

Hence $T_1^{\#}(I) \circ \pi(x) = \pi(T_1 x)$. Similarly, $T_2^{\#}(I) \circ \pi(x) = \pi(T_1 x)$. Therefore (1) is proved. Similarly we can prove (2).

(3) Let $N = T_1^{\#}(I) - T_2^{\#}(I)$. Then

$$\begin{aligned}N(f \circ x) &= \pi(x)(N \circ f) = (\pi(x) \circ N)(f) \\ &= \big(\pi(x) \circ T_1^{\#}(I) - \pi(x) \circ T_2^{\#}(I)\big)(f) \\ &= \big(\pi(T_2 x) - \pi(T_2 x)\big)(f) = 0.\end{aligned}$$

Similarly, $N(x \circ' f) = 0$. Therefore $N \in N^{**}$ and so $T_1^{\#}(I) = T_2^{\#}(I) + N$.

(4) Since $T_1^{\#}(I)$ and $T_2^{\#}(I) \in N^{**}$, we have

$$\pi(T_1 x)(f) = \big(T_2^{\#}(I) \circ \pi(x)\big)(f) = T_2^{\#}(I)(x \circ' f) = 0.$$

Hence $T_1 x = 0$ and so $T_1 = 0$. Also

$$\pi(T_2 x)(f) = \big(\pi(x) \circ T_1^{\#}(I)\big)(f) = T_1^{\#}(I)(f \circ x) = 0.$$

Therefore $T_2 = 0$ and so $T = 0$. This completes the proof of the lemma.

Notation. In the rest of this paper, for each $F \in M^{**}$, we write $\tilde{F} = F + N^{**}$.

5. The algebras $M(A)$ and M^{**}/N^{**}. We have the main result of this paper.

THEOREM 5.1. *Let A be a semisimple Banach algebra with an approximate identity $\{e_\alpha\}$. Then $M(A)$ is isometrically isomorphic to M^{**}/N^{**}.*

PROOF. Let $T = (T_1, T_2) \in M(A)$. Then by Lemma 4.3, $T_1^{\#}(I)$ and $T_2^{\#}(I) \in M^{**}$ and $\widetilde{T_1^{\#}(I)} = \widetilde{T_2^{\#}(I)}$. We define a mapping Φ from $M(A)$ to M^{**}/N^{**} by

$$\Phi(T) = \widetilde{T_1^{\#}(I)} = \widetilde{T_2^{\#}(I)}.$$

We show that Φ is an isometric isomorphism from $M(A)$ onto M^{**}/N^{**}. It is clear that Φ is linear. Let $x \in A$ and $f \in A^*$. If $\Phi(T) = 0$, then $T_1^{\#}(I)$ and $T_2^{\#}(I)$ are both in N^{**}. Hence by Lemma 4.3, $T = 0$. Therefore Φ is one-one. Let $S = (S_1, S_2) \in M(A)$. Since $ST = (S_1T_1, T_2S_2)$, by Lemma 4.3, we have

$$\begin{aligned}\big(S_1^{\#}(I) \circ T_1^{\#}(I)\big)(f \circ x) &= \big(\pi(x) \circ S_1^{\#}(I) \circ T_1^{\#}(I)\big)(f) \\ &= \big(\pi(S_2x) \circ T_1^{\#}(I)\big)(f) = \pi(T_2S_2x)(f) \\ &= \big(\pi(x) \circ (S_1T_1)^{\#}(I)\big)(f) = (S_1T_1)^{\#}(I)(f \circ x).\end{aligned}$$

Similarly, we have

$$\big(S_1^{\#}(I) \circ T_1^{\#}(I)\big)(x \circ' f) = (S_1T_1)^{\#}(I)(x \circ' f).$$

Therefore $\Phi(ST) = \Phi(S)\Phi(t)$ and so Φ is an isomorphism. Let $F \in M^{**}$. Define

$$P_1(x) = F \circ \pi(x) \quad \text{and} \quad P_2(x) = \pi(x) \circ F \qquad (x \in A).$$

Since $F \in M^{**}$, $P = (P_1, P_2) \in M(A)$. Also

$$\begin{aligned}P_1^{\#}(I)(f \circ x) &= I\big((f \circ x) \circ P_1\big) = \lim_{\alpha} \big((f \circ x) \circ P_1\big)(e_\alpha) \\ &= \lim_{\alpha} f\big(x(F \circ \pi(e_\alpha))\big) = \lim_{\alpha} f\big((\pi(x) \circ F)e_\alpha\big) \\ &= \lim_{\alpha} f\big(e_\alpha(\pi(x) \circ F)\big) = \lim_{\alpha} f\big(\pi(e_\alpha x) \circ F\big) \\ &= \lim_{\alpha} \big(\pi(e_\alpha x) \circ F\big)(f) = \lim_{\alpha} \pi(e_\alpha x)(F \circ f) \\ &= (F \circ f)(x) = F(f \circ x).\end{aligned}$$

Similarly, we have $P_1^{\#}(I)(x \circ' f) = F(x \circ' f)$. Therefore $\Phi(P) = \widetilde{P_1^{\#}} = \tilde{F}$. Hence Φ is an onto mapping. It is easy to see that $\|\Phi(T)\| \leqslant \|T\|$. Since $\|T_1\| = \sup\{|f(T_1x)|: \|x\| \leqslant 1,\ \|f\| \leqslant 1$ for all $x \in A$ and $f \in A^*\}$, for given $\varepsilon > 0$, there exists $y \in A$ and $g \in A^*$ with $\|y\| \leqslant 1$ and $\|g\| \leqslant 1$ such that $|g(T_1y)| \geqslant \|T_1\| - \varepsilon$. Then

$$\begin{aligned}I\big((y \circ' g) \circ T_1\big) &= \lim_{\alpha} (y \circ' g)(T_1e_\alpha) = \lim_{\alpha} g\big((T_1e_\alpha)y\big) = \lim_{\alpha} g\big(T_1(e_\alpha y)\big) \\ &= \lim_{\alpha} (g \circ T_1)(e_\alpha y) = (g \circ T_1)(y) = g(T_1y).\end{aligned}$$

Hence for all $N \in N^{**}$, we have

$$\begin{aligned}\|T_1^{\#}(I) + N\| &\geqslant \big|\big(T_1^{\#}(I) + N\big)(y \circ' g)\big| = \big|T_1^{\#}(I)(y \circ' g)\big| \\ &= \big|I\big((y \circ' g) \circ T_1\big)\big| = |g(T_1y)| \geqslant \|T_1\| - \varepsilon.\end{aligned}$$

Since ε is arbitrary, we have

$$\|\Phi(T)\| = \big\|\widetilde{T_1^{\#}(I)}\big\| \geqslant \|T_1\|.$$

Similarly, we can show that $\|\Phi(T)\| \geqslant \|T_2\|$ and so $\|\Phi(T)\| \geqslant \|T\|$. Therefore $\|\Phi(T)\| = \|T\|$. This completes the proof of the theorem.

COROLLARY 5.2. *Let A be as in Theorem 5.1. If $(A^{**}, \circ)$ has an identity I, then $N^{**} = (0)$ and so $M(A)$ is isometrically isomorphic to M^{**}.*

PROOF. Let $N \in N^{**}$. Then for all $f \in A^*$, we have

$$N(f) = (I \circ N)(f) = \lim_{\alpha} (\pi(e_\alpha) \circ N)(f) = \lim_{\alpha} N(f \circ e_\alpha) = 0.$$

Therefore $N^{**} = (0)$ and the corollary follows from Theorem 5.1.

COROLLARY 5.3. *Let A be as in Theorem* 5.1. *If A is Arens regular, then $M(A)$ is isometrically isomorphic to M^{**}.*

PROOF. Since A is Arens regular, $(A^{**}, \circ)$ has an identity I. Therefore the result follows from Corollary 5.2.

We now have a slight improvement of [**10**, Theorem, p. 442].

THEOREM 5.4. *Let A be a semisimple Banach algebra. Then $M(A)$ is isometrically isomorphic to $(A^{**}, \circ)$ if and only if A has the following properties*:

(1) *A is Arens regular.*

(2) *A has an approximate identity.*

(3) *$\pi(A)$ is a two-sided ideal of $(A^{**}, \circ)$.*

PROOF. Suppose that $M(A)$ is isometrically isomorphic to $(A^{**}, \circ)$. Then by Theorem 5.1, we have $N^{**} = (0)$ and $M^{**} = A^{**}$. Hence $\pi(A)$ is a two-sided ideal of $(A^{**}, \circ)$. By Theorem 4.2, A is Arens regular and $(A^{**}, \circ)$ has an identity. Hence it follows from [**3**, Proposition 7, p. 147] that A has an approximate identity. Therefore A has properties (1), (2), and (3).

Conversely, suppose that A has properties (1), (2), and (3). Then $M^{**} = A^{**}$ and so by Corollary 5.3, $M(A)$ is isometrically isomorphic to $(A^{**}, \circ)$. This completes the proof of the theorem.

6. Banach *-algebras. Let A be a Banach *-algebra with a continuous involution. For all $x \in A$, $f \in A^*$, and $F \in A^{**}$, we define

$$f^*(x) = \overline{f(x^*)} \quad \text{and} \quad F^*(f) = \overline{F(f^*)},$$

where the bar denotes the complex conjugation. Then $f^* \in A$ and $F^* \in A^{**}$. If A is a B^*-algebra, then $(A^{**}, \circ)$ is a B^*-algebra under the involution $F \to F^*$ (see [**7**, p. 192]).

LEMMA 6.1. *Let A be a Banach *-algebra with a continuous involution. Then*

(1) *For all F and $G \in A^{**}$, we have*

$$(F \circ G)^* = G^* \circ' F^* \quad \textit{and} \quad (F \circ' G)^* = G^* \circ F^*.$$

(2) *A is Arens regular if and only if $(A^{**}, \circ)$ is a *-algebra.*

PROOF. (1) For all $x \in A$ and $F \in A^*$, it is easy to show that $f^* \circ x = (x^* \circ' f)^*$ and so $G \circ f^* = (f \circ' G^*)^*$. Then

$$\begin{aligned}(F \circ G)^*(f) &= \overline{F(G \circ f^*)} = \overline{F\big((f \circ' G^*)^*\big)} \\ &= F^*(f \circ' G^*) = (G^* \circ' F^*)(f).\end{aligned}$$

Hence $(F \circ G)^* = G^* \circ' F^*$. Similarly, we have $(F \circ' G)^* = G^* \circ F^*$.

(2) If A is Arens regular, then it follows from (1) that $(A^{**}, \circ)$ is a *-algebra. Conversely, suppose that $(A^{**}, \circ)$ is a *-algebra. Then

$$F \circ G = (F \circ G)^{**} = (G^* \circ F^*)^* = F^{**} \circ' G^{**} = F \circ' G.$$

Hence A is Arens regular and the lemma is proved.

If A is a semisimple Banach *-algebra and $T = (T_1, T_2) \in M(A)$, then $T = (T_1, T_2) \to T^* = (T_1^*, T_2^*)$ is a continuous involution on $M(A)$, where $T_i^*(x) = (T_i(x^*))^*$ $(i = 1, 2)$.

THEOREM 6.2. *Let A be a semisimple Banach *-algebra. If A is Arens regular and A has an approximate identity $\{e_\alpha\}$ with $e_\alpha^* = e_\alpha$, then $M(A)$ is isometrically *-isomorphic to M^{**}.*

PROOF. By Corollary 5.3, $M(A)$ is isometrically isomorphic to M^{**}. By Lemma 6.1, $(A^{**}, \circ)$ is a *-algebra and so M^{**} is a closed *-subalgebra of $(A^{**}, \circ)$. Let $T = (T_1, T_2) \in M(A)$. Then, for all $f \in A^*$, we have

$$\begin{aligned}(T_1^*)^{\#}(I)(f) &= \lim_\alpha (f \circ T_1^*)(e_\alpha) = \lim_\alpha f(T_1^* e_\alpha) \\ &= \lim_\alpha f((T_1(e_\alpha))^*) = \lim_\alpha \overline{(f^* \circ T_1)(e_\alpha)} = \overline{I(f^* \circ T_1)} \\ &= T_1^{\#}(I)(f^*) = (T_1^{\#}(I))^*(f).\end{aligned}$$

Hence $(T_1^*)^{\#}(I) = (T_1^{\#}(I))^*$. Therefore $\Phi(T^*) = (\Phi(T))^*$. This completes the proof of the theorem.

If A is a B^*-algebra, then A is Arens regular and A has an approximate identity. Therefore we have the following result:

COROLLARY 6.3. *If A is a B^*-algebra, then $M(A)$ is isometrically *-isomorphic to M^{**}.*

REMARK. Corollary 6.3 is [**8**, Lemma 2.1, p. 80].

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