ARENS PRODUCT AND THE ALGEBRA OF DOUBLE MULTIPLIERS. II

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Abstract. We show that the algebra of double multipliers of a certain Banach algebra $A$ can be embedded in the second conjugate space $A^{**}$ of $A$. This generalizes the previous work by the author for $B^*$-algebras.

1. Introduction. Let $A$ be a semisimple Banach algebra and $M(A)$ the algebra of double multipliers on $A$. The author recently showed that $M(A)$ is isomorphic to $(A^{**}, °)$ if and only if $A$ has the following properties: (1) $A$ is Arens regular, (2) $A$ has an approximate identity, and (3) $\pi(A)$ is an ideal of $(A^{**}, °)$ [10, Theorem, p. 442]. In general, not many Banach algebras possess all these properties. The purpose of this paper is to establish an embedding of $M(A)$ in the second conjugate space $A^{**}$ of $A$ for a semisimple Banach algebra $A$ with an approximate identity. In fact, we show that $M(A)$ is isometrically isomorphic to $M^{**}/N^{**}$, where $M^{**} = \{F \in A^{**} : F^\circ \pi(x) = \pi(x) \circ F \in \pi(A) \text{ for all } x \in A\}$ and $N^{**} = \{F \in A^{**} : F(f \circ \pi) = F(x \circ' f) = 0 \text{ for all } x \in A \text{ and } f \in a^* \}$. The main idea of the proof of this result is essentially contained in the proofs of [8, Lemma 2.1, p. 80 and 10, Theorem, p. 422]. This embedding was studied in [1 and 8] for $B^*$-algebras.

2. Notation and preliminaries. Definitions not explicitly given are taken from Rickart's book [6].

Let $A$ be a Banach algebra. Then $A^*$ and $A^{**}$ will denote the first and second conjugate spaces of $A$, and $\pi$ the canonical map of $A$ into $A^{**}$. The two Arens products on $A^{**}$ are defined in stages according to the following rules (see [2]). Let $x, y \in A$, $f \in A^*$, and $F, G \in A^{**}$.

- Define $f \circ x$ by $(f \circ x)(y) = f(xy)$. Then $f \circ x \in A^*$.
- Define $G \circ f$ by $(G \circ f)(x) = G(f(x))$. Then $G \circ f \in A^*$.
- Define $F \circ G$ by $(F \circ G)(f) = F(G \circ f)$. Then $F \circ G \in A^{**}$.
- Define $x \circ' f$ by $(x \circ' f)(y) = f(yx)$. Then $x \circ' f \in A^*$.
- Define $f \circ' F$ by $(f \circ' F)(x) = F(x \circ' f)$. Then $f \circ' F \in A^*$.
- Define $F \circ' G$ by $(F \circ' G)(f) = G(f \circ' F)$. Then $F \circ' G \in A^{**}$.

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$A^{**}$ is a Banach algebra under the products $F \circ G$ and $F \circ' G$ and $\pi$ is an algebra isomorphism of $A$ into $(A^{**}, \circ)$ and $(A^{**}, \circ')$. In general, $\circ$ and $\circ'$ are distinct on $A^{**}$. If they agree on $A^{**}$, then $A$ is called Arens regular.

The following properties of the Arens products will be used often in the rest of this paper.

**Lemma 2.1.** Let $A$ be a Banach algebra. Then for all $x \in A$, $f \in A^*$, and $F, G \in A^{**}$, we have

1. $\pi(x) \circ F = \pi(x) \circ' F$ and $F \circ \pi(x) = F \circ' \pi(x)$.
2. If $\{F_\beta\} \subset A^{**}$ and $F_\beta \to F$ weakly in $A^{**}$, then $F_\beta \circ G \to F \circ G$ and $G \circ' F_\beta \to G \circ' F$ weakly.
3. $F(x^* f) = (F \circ \pi(x))(f)$.

**Proof.** (1) is proved in [2, p. 843] and (2) in [2, p. 842].

(3) It is easy to see that $\pi(x) \circ f = x \circ' f$. Therefore

$$(F \circ \pi(x))(f) = F(\pi(x) \circ f) = F(x \circ' f).$$

This completes the proof of the lemma.

We say that a Banach algebra $A$ has an approximate identity if there exists a net $(e_\alpha)$ in $A$ such that $\|e_\alpha\| \leq 1$ for all $\alpha$ and $x = \lim e_\alpha x = \lim x e_\alpha$ for all $x \in A$.

Let $A$ be a semisimple Banach algebra. A pair $(T_1, T_2)$ of operators from $A$ to $A$ is called a double multiplier (centralizer) on $A$ provided that $x(T_1 y) = (T_2 x) y$ for all $x, y \in A$. It is knowns that $T_1$ and $T_2$ are continuous linear operators on $A$ such that $T_1(x y) = (T_1 x) y$ and $T_2(x y) = x T_2 y$. The set $M(A)$ of all double multipliers on $A$ is a Banach algebra with identity and $A$ can be identified as a two-sided ideal of $M(A)$ (see [4 and 5]).

In this paper, all algebras and spaces under consideration are over the complex field.

**3. Extensions of double multipliers.** Let $A$ be a semisimple Banach algebra. We show that each double multiplier $T = (T_1, T_2)$ on $A$ can be extended to a double multiplier on $(A^{**}, \circ)$. In fact, for all $x \in A$ and $f \in A^*$, we define

$$(f \circ T_1)(x) = f(T_1 x) \quad \text{and} \quad (f \circ T_2)(x) = f(T_2 x).$$

Then $f \circ T_1$ and $f \circ T_2 \in A^*$. For all $F \in A^{**}$, we define

$$T_1^#(F)(f) = F(f \circ T_1) \quad \text{and} \quad T_2^#(F)(f) = F(f \circ T_2).$$

Then $T_1^#(f)$ and $T_2^#(F) \in A^{**}$.

**Theorem 3.1.** Let $A$ be a semisimple Banach algebra. Then for each $T = (T_1, T_2) \in M(A)$, $T^# = (T_1^#, T_2^#)$ is a continuous double multiplier on $(A^{**}, \circ)$ and $T^# \mid \pi(A) = T$. 
Proof. Let $F$ and $G \in A^{**}$. Then by Goldstine’s Theorem, there exist nets $\{x_m\}$ and $\{y_n\}$ in $A$ such that $\pi(x_m) \to F$ and $\pi(y_n) \to G$ weakly in $A^{**}$. Then for all $f \in A^*$, we have

$$(F \circ T_1^*(G))(f) = \lim_m (T_1^*(G) \circ f)(x_m) = \lim_m G((f \circ x_m) \circ T_1) = \lim_m \lim_n ((f \circ x_m) \circ T_1)(y_n) = \lim_n f((T_2 x_m) y_n) = \lim_m G(f \circ T_2 x_m) = \lim_m ((G \circ f) \circ T_2)(x_m) = (T_2^*(F) \circ G)(f).$$

Therefore $F \circ T_1^*(G) = T_2^*(F) \circ G$. Similarly, we can show that $T_1^*(F \circ G) = T_1^*(F) \circ T_2^*(G)$ and $T_2^*(F \circ G) = F \circ T_2^*(G)$. It is easy to see that $\|T_1^*\| \leq \|T_1\|$ and $\|T_2^*\| \leq \|T_2\|$. Therefore $\|T^*\| \leq \|T\|$. Since $T_1^*(\pi(x)) = \pi(T_1 x)$ and $T_2^*(\pi(x)) = \pi(T_2 x)$, we have $T^* | \pi(A) = T$ and the theorem is proved.

4. The subalgebras $M^{**}$ and $N^{**}$ of $(A^{**}, \circ)$. For any Banach algebra $A$, set $M^{**} = \{F \in A^{**}: F \circ \pi(x) \text{ and } \pi(x) \circ F \in \pi(A) \text{ for all } x \in A\}$ and

$N^{**} = \{F \in A^{**}: F(f \circ x) = F(x \circ f) = 0 \text{ for all } x \in A \text{ and } f \in A^*\}.$

Then $\pi(A)$ is a two-sided ideal of $(A^{**}, \circ)$ if and only if $M^{**} = A^{**}$.

Lemma 4.1. Let $A$ be a Banach algebra. Then $M^{**}$ and $N^{**}$ are closed subalgebras of $(A^{**}, \circ)$ and $(A^{**}, \circ')$. Also $N^{**}$ is a two-sided ideal in $M^{**}$ and $N^{**} = \{F \in A^{**}: A^{**} \circ F = F \circ A^{**} = (0)\}$.

Proof. For any $F \in N^{**}$, easy calculations show that $\pi(A) \circ F = (0)$ and $F \circ \pi(A) = (0)$. Hence $A^{**} \circ F = F \circ A^{**} = (0)$ and so $N^{**} = \{F \in A^{**}: A^{**} \circ F = F \circ A^{**} = (0)\}$. Thus if $F, G \in N^{**}$, then $\pi(A) \circ (F \circ G) = (F \circ G) \circ \pi(A) = (0)$ and so $F \circ G \in N^{**}$. Similarly, $F \circ G \in N^{**}$. It is easy to check that $M^{**}$ and $N^{**}$ are closed subalgebras in $(A^{**}, \circ)$ and $(A^{**}, \circ')$ and $N^{**}$ is a two-sided ideal of $M^{**}$. Therefore the lemma is proved.

In the rest of this section, $A$ will be a semisimple Banach algebra with an approximate identity $\{e_\alpha\}$. It follows from [3, Proposition 7, p. 146] that $(A^{**}, \circ)$ has a right identity $I$. Also $I$ is a left identity for $(A^{**}, \circ')$.

Theorem 4.2. Let $A$ be a semisimple Banach algebra with an approximate identity $\{e_\alpha\}$. Then the Arens products $\circ$ and $\circ'$ agree on $M^{**}/N^{**}$ and $M^{**}/N^{**}$ is a Banach algebra with an identity.

Proof. Let $F, G \in M^{**}$, $f \in A^*$, and $x \in A$. Then by Cohen’s Factorization Theorem [3, Theorem 10, p. 61], $x = yz$ with $y, z \in A$. Hence

$$(F \circ G - F \circ' G)(f \circ x) = (F \circ G - F \circ' G)((f \circ y) \circ z) = (\pi(z) \circ F \circ G - \pi(z) \circ F \circ G)(f \circ y) = 0.$$
Similarly, \( (F \circ G - F' \circ G)(x \circ' f) = 0 \). Therefore \( F \circ G - F' \circ G \in N^{**} \) and so \( \circ \) and \( \circ' \) agree on \( M^{**}/N^{**} \). Since \( I \) is a right identity for \( (A^{**}, \circ) \) and a left identity for \( (A^{**}, \circ) \) and \( I \in M^{**} \), it follows that \( M^{**}/N^{**} \) is a Banach algebra with an identity.

**Remark.** For a certain Banach algebra \( A \), \( N^{**} \) is the radical of \( (A^{**}, \circ) \) (for example, see [9, Theorem 4.1, p. 440]).

The following result is essentially contained in the proof of [8, Lemma 2.1, p. 80]. It is useful in the next section.

**Lemma 4.3.** Let \( A \) be a semisimple Banach algebra with an approximate identity \( \{ e_{\alpha} \} \), \( T = (T_1, T_2) \in M(A) \), and \( x \in A \). Then

1. \( T_1^{#}(I) \circ \pi(x) = T_2^{#}(I) \circ \pi(x) = \pi(T_1x) \).
2. \( \pi(x) \circ T_1^{#}(I) = \pi(x) \circ T_2^{#}(I) = \pi(T_2x) \).
3. \( T_1^{#}(I) = T_2^{#}(I) + N \) for some \( N \in N^{**} \).
4. If \( T_1^{#}(I) \) and \( T_2^{#}(I) \) are both in \( N^{**} \), then \( T = 0 \).

**Proof.** (1) For all \( f \in A^{**} \), we have

\[
(T_1^{#}(I) \circ \pi(x))(f) = I((\pi(x) \circ f) \circ T_1) = \lim_{\alpha} ((\pi(x) \circ f) \circ T_1)(e_{\alpha})
\]

\[
= \lim_{\alpha} f(T_1(e_{\alpha}x)) = \lim_{\alpha} (f \circ T_1)(e_{\alpha}x)
\]

\[
= (f \circ T_1)(x) = f(T_1x).
\]

Hence \( T_1^{#}(I) \circ \pi(x) = \pi(T_1x) \). Similarly, \( T_2^{#}(I) \circ \pi(x) = \pi(T_1x) \). Therefore (1) is proved. Similarly we can prove (2).

(3) Let \( N = T_1^{#}(I) - T_2^{#}(I) \). Then

\[
N(f \circ x) = \pi(x)(N \circ f) = (\pi(x) \circ N)(f)
\]

\[
= (\pi(x) \circ T_1^{#}(I) - \pi(x) \circ T_2^{#}(I))(f)
\]

\[
= (\pi(T_2x) - \pi(T_2x))(f) = 0.
\]

Similarly, \( N(x \circ' f) = 0 \). Therefore \( N \in N^{**} \) and so \( T_1^{#}(I) = T_2^{#}(I) + N \).

(4) Since \( T_1^{#}(I) \) and \( T_2^{#}(I) \in N^{**} \), we have

\[
\pi(T_1x)(f) = (T_2^{#}(I) \circ \pi(x))(f) = T_2^{#}(I)(x \circ' f) = 0.
\]

Hence \( T_1x = 0 \) and so \( T_1 = 0 \). Also

\[
\pi(T_2x)(f) = (\pi(x) \circ T_1^{#}(I))(f) = T_1^{#}(I)(f \circ x) = 0.
\]

Therefore \( T_2 = 0 \) and so \( T = 0 \). This completes the proof of the lemma.

**Notation.** In the rest of this paper, for each \( F \in M^{**} \), we write \( \bar{F} = F + N^{**} \).

**5. The algebras \( M(A) \) and \( M^{**}/N^{**} \).** We have the main result of this paper.

**Theorem 5.1.** Let \( A \) be a semisimple Banach algebra with an approximate identity \( \{ e_{\alpha} \} \). Then \( M(A) \) is isometrically isomorphic to \( M^{**}/N^{**} \).

**Proof.** Let \( T = (T_1, T_2) \in M(A) \). Then by Lemma 4.3, \( T_1^{#}(I) \) and \( T_2^{#}(I) \in M^{**} \) and \( T_1^{#}(I) = T_2^{#}(I) \). We define a mapping \( \Phi \) from \( M(A) \) to \( M^{**}/N^{**} \) by

\[
\Phi(T) = \overline{T_1^{#}(I)} = \overline{T_2^{#}(I)}.
\]
We show that \( \Phi \) is an isometric isomorphism from \( M(A) \) onto \( M^{**}/N^{**} \). It is clear that \( \Phi \) is linear. Let \( x \in A \) and \( f \in A^* \). If \( \Phi(T) = 0 \), then \( T_1^*(I) \) and \( T_2^*(I) \) are both in \( N^{**} \). Hence by Lemma 4.3, \( T = 0 \). Therefore \( \Phi \) is one-one. Let \( S = (S_1, S_2) \in M(A) \). Since \( ST = (S_1T_1, T_2S_2) \), by Lemma 4.3, we have
\[
(S_1^*(I) \circ T_1^*(I))(f \circ x) = (\pi(x) \circ S_1^*(I) \circ T_1^*(I))(f) = (\pi(S_2x) \circ T_1^*(I))(f) = \pi(T_2S_2x)(f) = (\pi(x) \circ (S_1^*(I))^{**})(f) = (S_1T_1)^*(I)(f \circ x).
\]
Similarly, we have
\[
(S_1^*(I) \circ T_1^*(I))(x \circ f) = (S_1T_1)^*(I)(x \circ f).
\]
Therefore \( \Phi(ST) = \Phi(S)\Phi(T) \) and so \( \Phi \) is an isomorphism. Let \( F \in M^{**} \). Define
\[
P_1(x) = F \circ \pi(x) \quad \text{and} \quad P_2(x) = \pi(x) \circ F \quad (x \in A).
\]
Since \( F \in M^{**} \), \( P = (P_1, P_2) \in M(A) \). Also
\[
P_1^*(I)(f \circ x) = I((f \circ x) \circ P_1) = \lim_{\alpha} ((f \circ x) \circ P_1)(e_{\alpha}) = \lim_{\alpha} f(x(F \circ \pi(e_{\alpha}))) = \lim_{\alpha} f((\pi(x) \circ F)e_{\alpha}) = \lim_{\alpha} f(e_{\alpha}(\pi(x) \circ F)) = \lim_{\alpha} f(\pi(e_{\alpha}x) \circ F) = \lim_{\alpha} \pi(e_{\alpha}x)(F \circ f) = (F \circ f)(x) = F(f \circ x).
\]
Similarly, we have \( P_2^*(I)(x \circ f) = F(x \circ f) \). Therefore \( \Phi(P) = \hat{P}^{**} = \hat{F} \). Hence \( \Phi \) is an onto mapping. It is easy to see that \( \|\Phi(T)\| \leq \|T\| \). Since \( \|T_1\| = \sup\{|f(T_1x)| : \|x\| \leq 1, \|f\| \leq 1 \text{ for all } x \in A \text{ and } f \in A^* \} \), for given \( \varepsilon > 0 \), there exists \( y \in A \) and \( g \in A^* \) with \( \|y\| \leq 1 \) and \( \|g\| \leq 1 \) such that \( |g(T_1y)| \geq \|T_1\| - \varepsilon \). Then
\[
I((y \circ g) \circ T_1) = \lim_{\alpha} (y \circ g)(T_1e_{\alpha}) = \lim_{\alpha} g((T_1e_{\alpha})y) = \lim_{\alpha} g(T_1(e_{\alpha}y)) = \lim_{\alpha} (g \circ T_1)(e_{\alpha}y) = (g \circ T_1)(y) = g(T_1y).
\]
Hence for all \( N \in N^{**} \), we have
\[
\|T_1^*(I) + N\| \geq |(T_1^*(I) + N)(y \circ g)| = |T_1^*(I)(y \circ g)| = |I((y \circ g) \circ T_1)| = |g(T_1y)| \geq \|T_1\| - \varepsilon.
\]
Since \( \varepsilon \) is arbitrary, we have
\[
\|\Phi(T)\| = \|T_1^*(I)\| \geq \|T_1\|.
\]
Similarly, we can show that \( \|\Phi(T)\| \geq \|T_2\| \) and so \( \|\Phi(T)\| \geq \|T\| \). Therefore \( \|\Phi(T)\| = \|T\| \). This completes the proof of the theorem.

**Corollary 5.2.** Let \( A \) be as in Theorem 5.1. If \( (A^{**}, \circ) \) has an identity \( I \), then \( N^{**} = (0) \) and so \( M(A) \) is isometrically isomorphic to \( M^{**} \).
Proof. Let \( N \in N^{**} \). Then for all \( f \in A^* \), we have
\[
N(f) = (I \circ N)(f) = \lim_{\alpha} \left( \pi(e_\alpha) \circ N \right)(f) = \lim_{\alpha} N(f \circ e_\alpha) = 0.
\]
Therefore \( N^{**} = (0) \) and the corollary follows from Theorem 5.1.

Corollary 5.3. Let \( A \) be as in Theorem 5.1. If \( A \) is Arens regular, then \( M(A) \) is isometrically isomorphic to \( M^{**} \).

Proof. Since \( A \) is Arens regular, \( (A^{**}, \circ) \) has an identity \( I \). Therefore the result follows from Corollary 5.2.

We now have a slight improvement of [10, Theorem, p. 442].

Theorem 5.4. Let \( A \) be a semisimple Banach algebra. Then \( M(A) \) is isometrically isomorphic to \( (A^{**}, \circ) \) if and only if \( A \) has the following properties:
(1) \( A \) is Arens regular.
(2) \( A \) has an approximate identity.
(3) \( \pi(A) \) is a two-sided ideal of \( (A^{**}, \circ) \).

Proof. Suppose that \( M(A) \) is isometrically isomorphic to \( (A^{**}, \circ) \). Then by Theorem 5.1, we have \( N^{**} = (0) \) and \( M^{**} = A^{**} \). Hence \( \pi(A) \) is a two-sided ideal of \( (A^{**}, \circ) \). By Theorem 4.2, \( A \) is Arens regular and \( (A^{**}, \circ) \) has an identity. Hence it follows from [3, Proposition 7, p. 147] that \( A \) has an approximate identity. Therefore \( A \) has properties (1), (2), and (3).

Conversely, suppose that \( A \) has properties (1), (2), and (3). Then \( M^{**} = A^{**} \) and so by Corollary 5.3, \( M(A) \) is isometrically isomorphic to \( (A^{**}, \circ) \). This completes the proof of the theorem.

6. Banach \(*\)-algebras. Let \( A \) be a Banach \(*\)-algebra with a continuous involution. For all \( x \in A \), \( f \in A^* \), and \( F \in A^{**} \), we define
\[
f^*(x) = f(x^*) \quad \text{and} \quad F^*(f) = F(f^*),
\]
where the bar denotes the complex conjugation. Then \( f^* \in A \) and \( F^* \in A^{**} \). If \( A \) is a \( B^* \)-algebra, then \( (A^{**}, \circ) \) is a \( B^* \)-algebra under the involution \( F \to F^* \) (see [7, p. 192]).

Lemma 6.1. Let \( A \) be a Banach \(*\)-algebra with a continuous involution. Then
(1) For all \( F \) and \( G \in A^{**} \), we have
\[
(F \circ G)^* = G^* \circ F^* \quad \text{and} \quad (F \circ G)^* = G^* \circ F^*.
\]
(2) \( A \) is Arens regular if and only if \( (A^{**}, \circ) \) is a \(*\)-algebra.

Proof. (1) For all \( x \in A \) and \( F \in A^* \), it is easy to show that \( f^* \circ x = (x^* \circ f)^* \) and so \( G \circ f^* = (f^* \circ G^*)^* \). Then
\[
(F \circ G)^*(f) = \overline{F(G \circ f^*)} = \overline{F((f^* \circ G^*)^*)} = F^*(f \circ G^*) = (G^* \circ F^*)(f).
\]
Hence \( (F \circ G)^* = G^* \circ F^* \). Similarly, we have \( (F \circ G)^* = G^* \circ F^* \).
If $A$ is Arens regular, then it follows from (1) that $(A^{**}, \circ)$ is a $*$-algebra. Conversely, suppose that $(A^{**}, \circ)$ is a $*$-algebra. Then

$$F \circ G = (F \circ G)^{**} = (G^{*} \circ F^{*})^{*} = F^{**} \circ G^{**} = F^{*} \circ G.$$  

Hence $A$ is Arens regular and the lemma is proved.

If $A$ is a semisimple Banach $*$-algebra and $T = (T_1, T_2) \in M(A)$, then $T = (T_1, T_2) \rightarrow T^{*} = (T_2^{*}, T_1^{*})$ is a continuous involution on $M(A)$, where $T_i^{*}(x) = (T_i(x^{*}))^*$ ($i = 1, 2$).

**Theorem 6.2.** Let $A$ be a semisimple Banach $*$-algebra. If $A$ is Arens regular and $A$ has an approximate identity $\{e_a\}$ with $e_a^{*} = e_a$, then $M(A)$ is isometrically $*$-isomorphic to $M^{**}$.

**Proof.** By Corollary 5.3, $M(A)$ is isometrically isomorphic to $M^{**}$. By Lemma 6.1, $(A^{**}, \circ)$ is a $*$-algebra and so $M^{**}$ is a closed $*$-subalgebra of $(A^{**}, \circ)$. Let $T = (T_1, T_2) \in M(A)$. Then, for all $f \in A^{*}$, we have

$$T_1^{*}(I)(f) = \lim_{\alpha} (f \circ T_1^{*})(e_{\alpha}) = \lim_{\alpha} f(T_1^{*}e_{\alpha})$$

$$= \lim_{\alpha} f(T_1(e_{\alpha}^{*}))^{*} = \lim_{\alpha} \overline{(f \circ T_1)(e_{\alpha})} = \overline{I(f \circ T_1)}$$

$$= T_1^{*}(I)(f^{*}) = (T_1^{*}(I))^{*}(f).$$

Hence $(T_1^{*})^{*}(I) = (T_1^{*}(I))^{*}$. Therefore $\Phi(T^{*}) = (\Phi(T))^{*}$. This completes the proof of the theorem.

If $A$ is a $B^{*}$-algebra, then $A$ is Arens regular and $A$ has an approximate identity. Therefore we have the following result:

**Corollary 6.3.** If $A$ is a $B^{*}$-algebra, then $M(A)$ is isometrically $*$-isomorphic to $M^{**}$.

**Remark.** Corollary 6.3 is [8, Lemma 2.1, p. 80].

**References**