PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 100, Number 3, July 1987

MINIMALITY OF GEODESICS IN GRASSMANN MANIFOLDS

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Dedicated to L. A. Santaló on his 75th birthday

ABSTRACT. In the Grassmann manifold of an arbitrary C^*-algebra, the geodesics of length less than π are curves of minimal length.

The classical Grassmann manifolds $G_{q,n}$ of q-planes in n-dimensional spaces can be alternatively described by symmetries (i.e., selfadjoint square roots of the identity) leaving a q-plane fixed. Hence the geometry of $G_{q,n}$ can be studied by transferring concepts and results to the space of symmetries, which turns out to be an analytic (disconnected) submanifold of the space of $n \times n$ matrices.

If in this interpretation we choose the operator norm of matrices, $G_{q,n}$ is *not* a Riemannian manifold. However a natural connection arises and the theorem below implies that short geodesics in $G_{q,n}$ are minimal curves for the arclength derived from the operator norm. This works equally well in the infinite-dimensional setting where no Riemannian structure is in general available (similar to $\mathrm{tr}(AB^*)$ for example).

Let $\mathcal{A}$ be a C^*-algebra with identity 1. The *Grassmannian* $\mathrm{Gr}(\mathcal{A})$ of $\mathcal{A}$ is the set of "symmetries" of $\mathcal{A}$, i.e., the elements $s \in \mathcal{A}$ with $s^* = s$ and $s^2 = 1$. Equivalently, $\mathrm{Gr}(\mathcal{A})$ can be identified to the set of selfadjoint idempotents p if $s = 2p - 1$.

Given a symmetry $s = 2p - 1$ denote by $\mathrm{Proj}_s \colon \mathcal{A} \to \mathcal{A}$ the real-linear norm-bounded projection

$$\mathrm{Proj}_s(z) = (1-p)zp + pz^*(1-p).$$

Let T_s and N_s be the range and kernel of Proj_s. Observe that

(1) *$x \in T_s$ if and only if x is selfadjoint and anticommutes with s.*

Also define the analytic map $E_s \colon \mathcal{A} \to \mathcal{A}$ by

$$E_s(z) = e^{zs/2} s e^{-zs/2}.$$

The following is an explicit form of "close symmetries are conjugate":

(2) *Let s and r be symmetries with $|s - r| = d < 2$ and set $x = 2s\log(a|a|^{-1})$, where $a = (1 + sr)/2$. Then $x \in T_s$ and $E_s(x) = r$.*

From $\|1 - a\| = \|s(s-r)/2\| = d/2 < 1$ and $aa^* = ((s+r)/2)^2 = a^*a$, it follows that a is an invertible normal element. Then $u = a|a|^{-1} = |a|^{-1}a$ is unitary with $\|1 - u\| \leq d/2 < 1$ (again from $\|1 - a\| \leq d/2$ and the spectral theorem). Hence $w = \log u$ is (well defined if the principal branch of log is used and) skewsymmetric: $-w = w^*$. Further (since $sa = ar = (s+r)/2$ is selfadjoint) $sa = ar = a^*s = ra^*$ and therefore s and r commute with $|a|$. In particular

$$sus = sa|a|^{-1}s = a^*s|a|^{-1}s = a^*|a|^{-1} = u^{-1}$$

Received by the editors December 25, 1985 and in revised form April 1, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L05.

whence (recall that $s = s^{-1}$ and use the series of log, for example) $sws = -w$ or $x = 2sw = -2ws$. Thus $x = x^*$ and $sx = -xs$, and (1) implies $x \in T_s$. Also

$$E_s(x) = e^{xs/2}se^{-xs/2} = e^{-w}se^{w} = u^*su = |a|^{-1}a^*sa|a|^{-1} = |a|^{-1}sa^*a|a|^{-1} = r,$$

which completes the proof of (2).

Next consider the map $\Phi\colon T_s \oplus N_s \to \mathcal{A}$, $\Phi(x,y) = E_s(x)+y$. Clearly $\Phi(0,0) = s$ and the derivative of Φ at $x = 0, y = 0$ is the "identity" map $(h,k) \to h + k$ from $T_x \oplus N_s$ to $T_s + N_s = \mathcal{A}$. According to (2) above $\mathrm{Gr}(\mathcal{A})$ is locally the image of $T_s \oplus \{0\}$ under the local diffeomorphism Φ. Hence

(3) $\mathrm{Gr}(\mathcal{A})$ *is a real-analytic submanifold of* $\mathcal{A}$. *The inclusion* $\mathrm{Gr}(\mathcal{A}) \hookrightarrow \mathcal{A}$ *identifies the tangent space to* $\mathrm{Gr}(\mathcal{A})$ *at* s *with the real Banach subspace* T_s *of* $\mathcal{A}$.

The normal bundle $\{N_s\}$ of $\mathrm{Gr}(\mathcal{A})$ induces the *canonical connection* of $\mathrm{Gr}(\mathcal{A})$ defined by

$$D_x y = \mathrm{Proj}_s \left(\frac{d}{dt} y\,(c(t)) \Big|_{t=0} \right),$$

where $x \in T_s$, y is a tangent field near s, and the curve $c(t) \in \mathrm{Gr}(\mathcal{A})$ satisfies $c(0) = s$, $\dot{c}(0) = x$.

(4) *The geodesics through* $s \in \mathrm{Gr}(\mathcal{A})$ *have the form*

$$\gamma(t) = e^{txs/2}se^{-txs/2}$$

and in particular the exponential of the canonical connection is given by the maps E_s *above.*

It suffices to verify that $d\gamma/dt(0) = x$ and $\mathrm{Proj}_{\gamma(t)}\, d^2\gamma/dt^2(t) = 0$; both follow from routine calculations using $d\gamma/dt = e^{txs/2}xe^{-txs/2}$ and $d^2\gamma/dt^2 = -x^2\gamma = -\gamma x^2$. Observe that this implies

$$d^2\gamma/dt^2 + \|x\|^2\gamma = \gamma(\|x\|^2 - x^2). \tag{5}$$

REMARK. All these notions coincide with the usual ones in the case of $G_{q,n}$ (see, for example, [**W**]).

THEOREM. *Short geodesics in the Grassmann manifold of a* C^**-algebra have minimal length. More precisely, a geodesic of length less that* π *is shorter than any other curve joining its endpoints.*

PROOF. Suppose $\gamma(t) = e^{txs/2}se^{-txs/2}$. Since $x = x^*$ there is a state f on $\mathcal{A}$ such that $f(x^2) = \|x\|^2$ (for C^*-algebra concepts see, for example, [**A**]). Let $\rho\colon \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be the associated cyclic representation with cyclic vector $\xi \in \mathcal{H}$, so that $f(a) = \langle \rho(a)\xi, \xi\rangle$ for all $a \in \mathcal{A}$. In this situation ξ is an eigenvector of $\rho(x^2)$ with eigenvalue $\|x\|^2$. In fact, since $0 \le \rho(x^2) \le \|x\|^2$ there is a hermitian operator $H \in \mathcal{L}(\mathcal{H})$ with $\|x\|^2 - \rho(x^2) = H^2$. Then

$$\|h\xi\|^2 = \langle (\|x\|^2 - \rho(x^2))\xi, \xi\rangle = \|x\|^2 - f(x^2) = 0$$

or $H\xi = 0$ whence $\rho(x^2)\xi = \|x\|^2\xi$.

Consider the curve $g(t)$ in the unit sphere Σ of $\mathcal{H}$ defined by $g(t) = \rho(\gamma(t))\xi$. From (5) it follows that

$$d^2g/dt^2 + \|x\|^2 g = \rho(d^2\gamma/dt^2 + \|x\|^2\gamma)\xi = \rho(\gamma)\rho(\|x\|^2 - x^2)\xi = 0$$

so that g is a geodesic in Σ for the Riemannian structure induced by $\mathcal{H}$. Suppose $\mu(t)$ is another curve in $\mathrm{Gr}(\mathcal{A})$ joining $\mu(0) = \gamma(0)$ and $\mu(\tau) = \gamma(\tau)$ and let $m(t) = \rho(\mu(t))\xi$. The curve m in Σ joins $g(0)$ to $g(\tau)$ and since $\|dm/dt\| \leq \|d\mu/dt\|$, its length $L(m) = \int_0^\tau \|dm/dt\|\, dt$ does not exceed the length $L(\mu)$ of μ. Also $\|dg/dt\| = \|x\|^2 = \|d\gamma/dt\|$, so $L(g) = L(\gamma)$. Now if $L(\gamma) < \pi$ then g is a "short" geodesic in Σ, hence minimal, and therefore $L(\gamma) = L(g) \leq L(m) \leq L(\mu)$. This proves the theorem.

According to (2) if $\|s - r\| = d < 2$, then a geodesic $E_s(tx)$ joining $s = E_s(0)$ to $r = E_s(x)$ can be explicitly described; its length L is $L = \|(dE_s(tx)/dt)(0)\| = \|x\|$. Using the definition of x in (2)

$$\|x\| = \|2s \log a|a|^{-1}\| = 2\| \log a - (1/2) \log aa^*\| = \| \log a - \log a^*\|$$

and then

$$L = \| \log(1 + rs)/2 - \log(1 + sr)/2\|.$$

The estimate $\|1 - u\| \leq d/2$ (see (2)) gives in particular $L \leq 2 \arcsin(d/2)$.

Suppose that $E_s(ty)$ $(y \in T_s)$ also joins s to $r = E_s(y)$ and abbreviate $v = e^{ys/2}$. Then $sus = e^{sx/2} = u^*$ so $usu = s$ and similarly $vsv = s$. But $usu^* = vsv^* = r$ so $ru^2 = usu = vsv = rv^2$ and $u^2 = v^2$, i.e., $(e^{xs/2})^2 = (e^{ys/2})^2$. This shows that uniqueness of minimizing goedesics is tied to uniqueness of square roots and logarithms. It can be proved that

(6) *If* $\|s = r\| = d < 1$, *then the minimizing geodesic joining* s *and* r *is unique.*

We are grateful to Norberto Sardinas for several valuable suggestions. In particular (2) and (6) are due to him.

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