PICARD DIMENSIONS OF CLOSE TO ROTATIONALLY INVARIANT DENSITIES

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ABSTRACT. The purpose of this paper is to show that the Picard dimensions of a rotation-free density $P$ and a general density $Q$ on the punctured unit disk $0 < |z| < 1$ are equal to each other if $|P(z) - Q(z)| = O(|z|^{-2})$ as $z \to 0$.

Before stating our result we fix terminologies. We denote by $\Omega$ the punctured unit disk $0 < |z| < 1$ which is viewed as an end of the punctured sphere $0 < |z| \leq \infty$ so that the unit circle $|z| = 1$ is the relative boundary $\partial \Omega$ of $\Omega$ and the origin $z = 0$ is the ideal boundary $\partial \Omega$ of $\Omega$. By a density $P$ on $\Omega$ we mean a nonnegative locally Hölder continuous function $P$ on the closure $\bar{\Omega} = \Omega \cup \partial \Omega$ of $\Omega$ so that $P$ may or may not have a singularity at $z = 0$. With a density $P$ on $\Omega$ we associate the class $PP(\Omega; \partial \Omega)$ of nonnegative continuous functions $u$ on $\bar{\Omega}$ such that $u$ satisfies the elliptic equation
\[
\Delta u = Pu, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},
\]
on $\Omega$ and vanishes on $\partial \Omega$. We also denote by $PP_1(\Omega; \partial \Omega)$ the subclass of $PP(\Omega; \partial \Omega)$ consisting of functions $u$ with the normalization $u(1/2) = 1$.

The Choquet theorem (cf., e.g., [7]) yields that there exists a bijective correspondence $u \leftrightarrow \mu$ between the convex set $PP_1(\Omega; \partial \Omega)$ and the set of probability measures $\mu$ on the set $\text{ex. } PP_1(\Omega; \partial \Omega)$ of extreme points of $PP_1(\Omega; \partial \Omega)$ such that
\[
u = \int_{\text{ex. } PP_1(\Omega; \partial \Omega)} v \, d\mu(v).
\]
Thus the set $\text{ex. } PP_1(\Omega; \partial \Omega)$ is essential for the class $PP_1(\Omega; \partial \Omega)$, and the cardinal number $\#(\text{ex. } PP_1(\Omega; \partial \Omega))$ of $\text{ex. } PP_1(\Omega; \partial \Omega)$ is referred to as the Picard dimension of a density $P$ on $\Omega$ at the ideal boundary $\partial \Omega$ of $\Omega$, in short $\dim P$, i.e.
\[
\dim P = \#(\text{ex. } PP_1(\Omega; \partial \Omega)).
\]
If $\dim P = 1$, then we say that the Picard principle is valid for $P$. In the study of the Picard principle the density $P_0(z) = |z|^{-2}$ plays an important role. For example the Picard principle for the density $P_\lambda(z) = |z|^{-2+\lambda}$ is valid if and only if $\lambda \in [0, \infty)$ [3, 4], the Picard principle for a density $P$ is valid if $P(z) = O(|z|^{-2})$ as $z \to 0$ [2], and the Picard principle for a density $Q$ with $P(z) \leq Q(z) \leq P(z) + C|z|^{-2}$ ($z \in \Omega$) for a positive constant $C$ and a rotationally invariant density $P$, i.e. a density $P$ satisfying $P(z) = P(|z|)$ ($z \in \Omega$), is valid if the Picard principle for $P$ is valid [8].

The purpose of this paper is to show that the density $P_0(z)$ also plays an important

Received by the editors November 7, 1984 and, in revised form, May 14, 1986.
1980 Mathematics Subject Classification (1985 Revision). Primary 31A35; Secondary 30F25.
Key words and phrases. Picard dimension and principle, Martin boundary.
The author was supported in part by Grant-in-Aid for Scientific Research No. 59340007, Japanese Ministry of Education, Science, and Culture.

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0002-9939/87 $1.00 + .25 per page
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role in the study of Picard dimensions by establishing the following theorem which is more general than the above results:

**Theorem 1.** If \( P \) is a rotationally invariant density on \( \Omega \) and \( Q \) is a general density of \( \Omega \) with

\[
|P(z) - Q(z)| = O(|z|^{-2}) \quad \text{as } z \to 0,
\]

then we have \( \dim P = \dim Q \).

We have seen that the range \( \dim \mathcal{D} \) of the mapping \( P \mapsto \dim P \) from the set \( \mathcal{D} \) of densities on \( \Omega \) into the set of cardinal numbers consists of all positive integers, the infinite countable cardinal number \( \aleph_0 \), and the cardinal number \( \aleph \) of the continuum \([5, 6]\):

\[
\dim \mathcal{D} = [1, \aleph]
\]

if the continuum hypothesis is postulated. In particular the range \( \dim \mathcal{D}_r \) of the set \( \mathcal{D}_r \) of rotationally invariant densities on \( \Omega \) consists of 1 and \( \aleph \) \([3]\):

\[
\dim \mathcal{D}_r = \{1, \aleph\}.
\]

Then the above theorem shows that the range of the set of densities \( Q \) on \( \Omega \) with (1) for a rotationally invariant density \( P \) on \( \Omega \) also consists of 1 and \( \aleph \).

We will recall in §1 the proof of (3) according to Nakai \([3]\) and prove Theorem 1 in §3 by using the fundamental properties of "Martin kernels" for rotationally invariant densities \( P \) on \( \Omega \) with \( \dim P = \aleph \) given in §§1 and 2.

The author is very grateful to Professor M. Nakai for his helpful suggestions.

1. Picard dimensions of rotationally invariant densities. 1. Consider a rotationally invariant density \( P \) on \( \Omega \). The unique bounded solution \( e_\rho \) of \( \Delta u = Pu \) on \( \Omega \) with boundary values 1 on \( \partial \Omega \) is referred to as the \( P \)-unit on \( \Omega \). We also consider rotationally invariant densities \( P_n \) \( (n = 0, 1, \ldots) \) on \( \Omega \) defined by \( P_n(z) = P(z) + n^2|z|^{-2} \) \((z \in \Omega)\) and denote by \( e_n \) the \( P_n \)-unit on \( \Omega \), where we follow the convention \( P_0 = P \) and \( e_0 = e_p \). Then \( e_n \) is positive and rotationally invariant on \( \Omega \) and the function \( e_n(r) \) of \( r \) in \((0,1]\) is the unique bounded solution of

\[
\int_0^r r^2 \phi''(r) + \frac{1}{r} \frac{d}{dr} \phi(r) - P_n(r)\phi(r) = 0 \quad (n = 0, 1, \ldots)
\]

on \((0,1]\) with boundary values 1 at \( r = 1 \). Since \( e_n(r)/e_0(r) \) is positive and increasing on \((0,1]\), there exists the limit

\[
\alpha_n(P) = \lim_{r \to 0} \frac{e_n(r)}{e_0(r)} \quad (n = 1, 2, \ldots)
\]

which is referred to as the \( n \)th singularity index of \( P \) at \( \partial \Omega \). In particular, we denote by \( \alpha(P) \) the first singularity index \( \alpha_1(P) \) and call it simply the singularity index of \( P \) at \( \partial \Omega \). Then we have the following fundamental inequality \([3]\):

\[
0 \leq \alpha(P)^{2^n-1/2} \leq \alpha_n(P) \leq \alpha(P)^n < 1 \quad (n = 1, 2, \ldots).
\]

We also have the monotoneity of singularity indices \([1]\): for another rotationally invariant density \( R \) on \( \Omega \)

\[
P \leq R \quad \text{implies} \quad \alpha(P) \leq \alpha(R).
\]
2. We denote by $G_P(z, \zeta)$ the $P$-Green’s function on $\Omega$, i.e. the Green’s function on $\Omega$ with respect to $\Delta u = Pu$, and consider the function

$$L_P(z, \zeta) = \frac{G_P(z, \zeta)}{\epsilon_P(\zeta)} \quad (z, \zeta \in \Omega).$$

In the case that $\alpha(P) = 0$ the function $L_P(z, \zeta)$ of $z$ converges to a minimal function $L_P(z)$ in $PP(\Omega; \partial \Omega) - \{0\}$ as $\zeta \to 0$ uniformly on every compact subset of $\Omega$, and hence we have

$$\text{ex. } PP_1(\Omega; \partial \Omega) = PP_1(\Omega; \partial \Omega) = \{L_P(z)/L_P(1/2)\}.$$

In the case that $\alpha(P) > 0$ the function $L_P(z, \zeta)$ of $z$ converges to a minimal function $L_P(z, \sigma)$ ($\sigma \in [0, 2\pi]$) in $PP(\Omega; \partial \Omega) - \{0\}$ as $|\zeta| \to 0$ and $\arg \zeta \to \sigma$ uniformly on every compact subset of $\Omega$. Furthermore, minimal functions $L_P(z, \sigma)$ are pairwise nonproportional, and

$$\text{ex. } PP_1(\Omega; \partial \Omega) = \{L_P(z, \sigma)/L_P(1/2, \sigma); \sigma \in [0, 2\pi]\},$$

where we remark that

$$L_P(z, \sigma) = L_P(ze^{-i\sigma}, 0) \quad (z \in \Omega; \sigma \in [0, 2\pi]),$$

$$L_P(z, 0) = L_P(\bar{z}, 0) \quad (z \in \Omega).$$

We then have the following characterization [3]:

$$\dim P = 1 + \alpha(P) \cdot N.$$

2. **Fundamental properties of $L_P(\cdot, 0)$**. 1. In this section we consider a rotationally invariant density $P$ on $\Omega$ with $\alpha(P) > 0$ and describe the properties of the function $L_P(\cdot, 0)$. We consider the $P$-Martin kernel $K_P(z, \zeta) = G_P(z, \zeta)/G_P(1/2, \zeta)$ ($z, \zeta \in \Omega$). As $|\zeta|, \arg \zeta \to 0$ we obtain the $P$-Martin kernel $K_P(z, 0e^{it})$ with pole at $0e^{it}$. The latter can be represented as $K_P(z, 0e^{it}) = L_P(z, 0)/L_P(1/2, 0)$.

Let $s$ be a positive number with $s < 1$ and $\sigma, \tau$ be nonnegative numbers with $\sigma < \tau \leq \pi$. Then the function

$$G_P(z, s) - G_P(z, se^{(\sigma+i\tau)})$$

is the $P$-Green’s function on

$$\Omega_{\sigma, \tau} = \left\{|z| < 1: \frac{\sigma + \tau}{2} - \pi < \arg z < \frac{\sigma + \tau}{2}\right\}$$

with pole at $s$ so that we have

$$G_P(z, s) \geq G_P(z, se^{(\sigma+i\tau)})$$

and hence

$$L_P(z, 0) \geq L_P(z, \sigma + \tau)$$

for any $z$ in $\Omega_{\sigma, \tau}$. Since $\sigma < \tau$, the point $re^{i\sigma}$ ($r \in (0, 1)$) is contained in $\Omega_{\sigma, \tau}$. Setting $z = re^{i\sigma}$ in the above inequality, and using (6) yields $L_P(re^{i\sigma}, 0) \geq L_P(re^{-i\tau}, 0)$. Thus we obtain by (7)

$$L_P(re^{i\sigma}, 0) \geq L_P(re^{i\tau}, 0) \quad (r \in (0, 1], \ |\sigma| \leq |\tau| \leq \pi).$$
2. It is easy to see that the Fourier coefficient
\[ M_P(r) = \frac{1}{2\pi} \int_0^{2\pi} L_P(re^{i\theta}, 0) \, d\theta \]
of \( L_P(re^{i\theta}, 0) \) is a positive solution of \( l_0\phi(r) = 0 \) on \((0,1)\) with vanishing boundary values at \( r = 1 \). On the other hand the function
\[ E_P(r) = e_P(r) \int_r^1 \frac{ds}{se_P(s)^2} \]
is also a positive solution of \( l_0\phi(r) = 0 \) on \((0,1)\) with vanishing boundary values at \( r = 1 \). Then there exists a positive constant \( \lambda_P \) such that \( M_P(r) = \lambda_P E_P(r) \) and thus by (8)
\[ (9) \quad L_P(r,0) \geq \lambda_P E_P(r) \quad (r \in (0,1]). \]

3. **Proof of Theorem 1.** 1. As a special case of condition (1) we assume in §§3.1–3.3 that a rotationally invariant density \( P \) on \( \Omega \) and a general density \( Q \) on \( \Omega \) satisfy
\[ P(z) \leq Q(z) \leq P(z) + C|z|^{-2} \quad (z \in \Omega) \]
for a positive constant \( C \). When \( \alpha(P) = 0 \) it was shown in [8] that \( \dim P = \dim Q = 1 \). We therefore also assume in §§3.1–3.3 that \( \alpha(P) > 0 \). Then we show in §3.3 that \( \dim Q > N_0 \) by using auxiliary results in §§3.1–3.2. The proof of Theorem 1 will then be completed in §3.4.

Take an integer \( k \) with \( k^2 > C \) and consider the rotationally invariant density \( R = P_k \) on \( \Omega \), where as before
\[ P_k(z) = P(z) + k^2|z|^{-2} \quad (z \in \overline{\Omega}). \]
Then
\[ G_P(z, \zeta) \geq G_Q(z, \zeta) \geq G_R(z, \zeta), \quad e_P(\zeta) = e_0(\zeta) \geq e_R(\zeta) = e_k(\zeta) \]
\((z, \zeta \in \Omega)\), where \( G_Q \) and \( G_R \) are the \( Q \)-Green’s and the \( R \)-Green’s functions on \( \Omega \), respectively. If we set
\[ m_Q = \min\{G_Q(1/2, \zeta); |\zeta| = 1/4\}, \quad M_Q = \max\{G_Q(1/2, \zeta); |\zeta| = 1/4\}, \]
then the maximum principle yields
\[ M_Q \frac{e_P(\zeta)}{e_P(1/4)} \geq G_Q(1/2, \zeta) \geq m_Q \frac{e_R(\zeta)}{e_R(1/4)} \]
for any \( \zeta \) with \( 0 < |\zeta| < 1/4 \). Therefore the \( Q \)-Martin kernel
\[ K_Q(z, \zeta) = \frac{G_Q(z, \zeta)}{G_Q(1/2, \zeta)} \]
satisfies
\[ (10) \quad \frac{e_R(1/4)}{m_Q} \frac{e_P(\zeta)}{e_R(\zeta)} L_P(z, \zeta) \geq K_Q(z, \zeta) \geq \frac{e_P(1/4)}{M_Q} \frac{e_R(\zeta)}{e_P(\zeta)} L_R(z, \zeta) \]
\((z \in \Omega; \; 0 < |\zeta| < 1/4)\).

2. Consider the Martin compactification \( \Omega^*_Q \) of \( \Omega \) with respect to the equation \( \Delta u = Qu \) and the ideal boundary \( \Gamma = \Gamma_Q = \Omega^*_Q - \Omega \). We denote by \( \Gamma(\sigma) \) (\( \sigma \in [0, 2\pi) \)) the set of ideal boundary points \( \zeta^* \) such that there exists a sequence \( \{\zeta_n\}_{n=1}^{\infty} \)
in $\Omega$ with $\lim |\varsigma_n| = 0$, $\lim \arg \varsigma_n = \sigma$, and $\lim \varsigma_n = \varsigma^*$. Then $\Gamma$ is divided into the family $\{\Gamma(\sigma)\}: \Gamma(\sigma) \neq \emptyset$, $\Gamma = \bigcup \{\Gamma(\sigma); \sigma \in [0,2\pi)\}$, with
\begin{equation}
\Gamma(\sigma) \cap \Gamma(\tau) = \emptyset \quad (\sigma \neq \tau).
\end{equation}
The above first and second properties are trivial and the last property (11) can be proved as follows:

Let $\varsigma^*$ be any ideal boundary point in $\Gamma(\sigma)$ and let $\{\varsigma_n\}_{n=1}^{\infty}$ be a corresponding sequence in $\Omega$ as above. Then $K_Q(z, \varsigma_n)$ converges to the Q-Martin kernel $K_Q(z, \varsigma^*)$ with pole at $\varsigma^*$ as $n \to \infty$ uniformly on every compact subset of $\Omega$ and $e_R(\varsigma_n)/e_P(\varsigma_n)$ converges to $\alpha_k(P)$ as $n \to \infty$. Further $L_R(z, \varsigma_n)$ converges to $L_R(z, \sigma)$ as $n \to \infty$ uniformly on every compact subset of $\Omega$ since $\alpha(R) > 0$ by (5). Therefore by (10) $K(z, \varsigma^*)$ satisfies
\begin{equation}
\frac{e_R(1/4)}{m_Q} \frac{1}{\alpha_k(P)} L_P(z, \sigma) \geq K_Q(z, \varsigma^*) \geq \frac{e_P(1/4)}{M_Q} \alpha_k(P) L_R(z, \sigma)
\end{equation}
($z \in \Omega; \varsigma^* \in \Gamma(\sigma); \sigma \in [0,2\pi)$), where by (4) $\alpha_k(P) > 0$. Contrary to the assertion we suppose $\Gamma(\sigma) \cap \Gamma(\tau) = \emptyset$ for some $\sigma$ and $\tau$ with $\sigma \neq \tau$. Then the inequalities (12) for $\varsigma^*$ in $\Gamma(\sigma) \cap \Gamma(\tau)$ yield
\begin{equation}
L_P(z, \sigma) \geq \beta L_R(z, \tau),
\end{equation}
where
\begin{equation}
\beta = \beta(P, Q, k) = \frac{e_P(1/4)}{e_R(1/4)} \frac{m_Q}{M_Q} \alpha_k(P)^2.
\end{equation}
Therefore by (6) and (9) we have $L_P(re^{i(\tau - \sigma)}, 0) \geq \beta \lambda_R E_R(r)$, and hence by (8)
\begin{equation}
L_P(re^{i \theta}, 0) \geq \beta \lambda_R E_R(r) \quad (r \in (0,1]; |\theta| \leq d(\sigma, \tau)),
\end{equation}
where $d(\sigma, \tau) = \min(|\tau - \sigma|, 2\pi - |\tau - \sigma|)$. Let $m = m(\sigma, \tau)$ be the minimum integer of the set of integers greater than $\pi/d(\sigma, \tau)$. Then by (6) and the inequalities
\begin{equation}
E_R(r) \geq \frac{e_k(r)}{e_0(r)} e_0(r) \int_{\tau}^{1} \frac{ds}{s e_0(s)^2} \geq \alpha_k(P) E_P(r)
\end{equation}
we have
\begin{equation}
\sum_{j=1}^{m} L_P \left( z, \frac{2(j-1)}{m} \pi \right) \geq \beta \lambda_R E_R(|z|) \geq \beta \lambda_R \alpha_k(P) E_P(|z|) \quad \text{for any } z \in \Omega.
\end{equation}
Since $E_P(|z|)$ is a function in $PP(\Omega; \partial \Omega)$ and every $L_P(\cdot, 2(j - 1)\pi/m)$ ($j = 1,2,\ldots,m$) is a minimal function in $PP(\Omega; \partial \Omega)$, there exist nonnegative constants $c_1,\ldots,c_m$ such that
\begin{equation}
E_P(|z|) = \sum_{j=1}^{m} c_j L_P \left( z, \frac{2(j-1)}{m} \pi \right).
\end{equation}
Thus we obtain by (6)
\begin{equation}
\sum_{j=1}^{m} c_j L_P \left( z, \frac{2(j-1)}{m} \pi \right) = \sum_{j=1}^{m} c_j L_P \left( ze^{-i\pi/m}, \frac{2(j-1)}{m} \pi \right)
= E_P(|z|) = \sum_{j=1}^{m} c_j L_P \left( z, \frac{2(j-1)}{m} \pi \right).
\end{equation}
This contradicts the fact that every $L_p(z, j\pi/m)$ ($j = 0, \ldots, 2m - 1$) is minimal.

3. We denote by $\Gamma_0$ the set of minimal points in $\Gamma = \Gamma_Q$. Then the Picard dimension of $Q$ is also given by the equality $\dim Q = \#\Gamma_0$ since
\[
\text{ex. } Q P_1(\Omega; \partial\Omega) = \{K_Q(\cdot, \zeta^*); \zeta^* \in \Gamma_0\}.
\]

Let $\sigma(\zeta^*)$ ($\zeta^* \in \Gamma_0$) be the unique number in $[0, 2\pi)$ with $\zeta^* \in \Gamma(\sigma(\zeta^*))$. If $\#\Gamma_0 \leq \aleph_0$, then there exist a number $\sigma_1$ in $[0, 2\pi) - \bigcup\{\zeta^*; \zeta^* \in \Gamma_0\}$ and a point $\zeta_1^*$ in $\Gamma(\sigma_1)$. The $Q$-Martin kernel $K_Q(\cdot, \zeta_1^*)$ with pole at $\zeta_1^*$ is represented in terms of $K_Q(\cdot, \zeta^*)$ ($\zeta^* \in \Gamma_0$):
\[
K_Q(\cdot, \zeta_1^*) = \sum_{\zeta^* \in \Gamma_0} c(\zeta^*) K_Q(\cdot, \zeta^*)
\]
for nonnegative constants $c(\zeta^*)$. Then the inequality
\[
K_Q(z, \zeta_1^*) \geq c(\zeta^*) K_Q(z, \zeta^*) \quad (z \in \Omega)
\]
is valid for any $\zeta^*$ in $\Gamma_0$ and thus by (12)
\[
L_p(z, \sigma_1) \geq \beta c(\zeta^*) L_R(z, \sigma(\zeta^*)) \quad (\zeta^* \in \Gamma_0, z \in \Omega).
\]

Now there exists a point $\zeta_2^*$ in $\Gamma_0$ with $c(\zeta_2^*) > 0$ and so $\sigma_1 \neq \sigma(\zeta_2^*)$. This together with inequality (13) for $\zeta^* = \zeta_2^*$ yield a contradiction by the similar argument that led to the proof of (11). We therefore have proved the following result:

**Lemma 1.** If $P$ is a rotationally invariant density on $\Omega$ and $Q$ is a general density on $\Omega$ with
\[
P(z) \leq Q(z) \leq P(z) + C|z|^{-2} \quad (z \in \Omega)
\]
for some positive constant $C$, then $\alpha(P) > 0$ implies $\dim Q > \aleph_0$.

4. We now assume that the rotationally invariant density $P$ on $\Omega$ and the general density $Q$ on $\Omega$ satisfy (1). Then there exists a positive constant $C$ such that
\[
P(z) - C|z|^{-2} \leq Q(z) \leq P(z) + C|z|^{-2} \quad (z \in \Omega).
\]
Consider the rotationally invariant density $R$ on $\Omega$ defined by
\[
R(z) = \max(P(z) - C|z|^{-2}, 0) \quad (z \in \overline{\Omega}).
\]
Then $R$ satisfies
\[
R(z) \leq P(z) \leq R(z) + C|z|^{-2},
\]
\[
R(z) \leq Q(z) \leq R(z) + 2C|z|^{-2} \quad (z \in \overline{\Omega}).
\]
Therefore, using Lemma 1, $\alpha(R) = 0$ implies $\dim P = \dim Q = 1$ and $\alpha(R) > 0$ implies $\dim P > \aleph_0$, $\dim Q > \aleph_0$. But by (3), $\dim P > \aleph_0$ implies $\dim P = \aleph$, and by (2), $\dim Q > \aleph_0$ implies $\dim Q = \aleph$ if the continuum hypothesis is postulated. Thus we conclude $\dim P = \dim Q$, and Theorem 1 is proved.

**References**


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