ON THE RESTRICTED MEAN VALUE PROPERTY

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ABSTRACT. Suppose that $u$ is continuous in the open unit disc and has the restricted mean value property. It is shown that if $u$ has finite boundary limits almost everywhere, and if $u$ possesses a harmonic majorant and minorant, the difference between which has finite radial upper limits everywhere, then $u$ is harmonic.

1. Introduction. A function $u(z)$ in the open unit disc $\Delta(0,1)$ has the restricted mean-value (rmv) property if for each $z \in \Delta(0,1)$ there is a positive number $\rho = \rho(z) < 1 - |z|$ such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} u(z + \rho e^{i\theta}) \, d\theta = u(z).$$

Here we shall prove

THEOREM 1. Suppose that $u(z)$ is continuous in $\Delta(0,1)$ and has the rmv property. If

(i) $\lim_{r \to 1} u(z)$ exists for almost all $\theta$, and

(ii) $u(z)$ has a harmonic majorant $h_2(z)$ and a harmonic minorant $h_1(z)$ such that for all $\theta \in [-\pi, \pi]$

$$\lim_{r \to 1} \{h_2(re^{i\theta}) - h_1(re^{i\theta})\} < \infty,$$

then $u$ is harmonic.

This is an extension of an earlier result of the author [1] in which instead of condition (ii) it is supposed the $u(z)$ is bounded. Both are motivated by problem 11 in J. E. Littlewood’s book [2]. It is a genuine extension since $h_2(z) - h_1(z)$ need not be bounded, as the following theorem shows.

THEOREM 2. Suppose that $h(z)$ is a nonnegative harmonic function in $\Delta(0,1)$ such that

$$\lim_{r \to 1} h(re^{i\theta}) < \infty$$

for every $\theta \in [-\pi, \pi]$. Then

$$h(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(t)re\{(e^{it} + z)/(e^{it} - z)\} \, dt,$$

where $H(t)$ is nonnegative and such that

$$\lim_{T \to 0} \frac{1}{2T} \int_{-T}^{T} H(t + \theta) \, dt < \infty$$

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for every \( \theta \in [-\pi, \pi] \). Conversely, if \( h(z) \) is defined by (1.3), where \( H(t) \) is nonnegative and satisfies (1.4) for every \( \theta \in [-\pi, \pi] \), then (1.2) holds for every \( \theta \in [-\pi, \pi] \).

As will be shown later, Theorem 2 has no natural counterpart for \( h^1 \) functions at least in the sense that there is an \( h^1 \) function \( h(z) \) of the form (1.3) satisfying

\[
\lim_{r \to 1} |h(re^{i\theta})| < \infty \quad \text{for every } \theta \in [-\pi, \pi].
\]

2. Two auxiliary functions. Given \( \zeta \in \Delta(0,1) \) let \( v_\zeta(z) \) be the harmonic function in \( \Delta(\zeta, \rho(\zeta)) \) which is the Poisson integral of the boundary values of \( u \).

Define

\[
\begin{align*}
V(z) &= \sup v_\zeta(z), \\
W(z) &= \inf v_\zeta(z),
\end{align*}
\]

the supremum and infimum extended over those \( \zeta \) such that \( |z - \zeta| < \rho(\zeta) \). From condition (ii) both \( V(z) \) and \( W(z) \) are finite for all \( z \) and

\[
h_2(z) \geq V(z) \geq v_\zeta(z) = u(z) \geq W(z) \geq h_1(z).
\]

These \( V \) and \( W \) are slightly simpler versions of functions introduced in [1]. The significant properties of \( V \) and \( W \) are

\[
(2.2) \quad V(z) \text{ is subharmonic and } W(z) \text{ is superharmonic},
\]

and

\[
(2.3) \quad \text{for a.a. } \theta \in [-\pi, \pi], \quad \lim_{z \to e^{i\theta}} V(z) = \lim_{z \to e^{i\theta}} W(z) = \lim_{z \to e^{i\theta}} u(z).
\]

To prove these we need

**Lemma.** Let \( \xi \in \Delta(0,1) \) and let \( z_n \to \xi \). If \( \zeta_n \) is such that \( |z_n - \zeta_n| < \rho(\zeta_n) \) for all \( n \) and if \( \rho(\zeta_n) - |z_n - \zeta_n| \to 0 \) as \( n \to \infty \), then

\[
(2.4) \quad v_{\zeta_n}(z_n) \to u(\xi).
\]

Suppose that (2.4) is false for some sequence \( \zeta_n \) (which may be assumed without loss to be convergent) with limit \( \zeta_0 \) say. Evidently \( \rho_0 \neq 0 \), where \( \rho_0 = \lim \rho(\zeta_n) \), so that \( \xi \) is a boundary point of \( \Delta(\zeta_0, \rho_0) \).

Given \( \varepsilon > 0 \) let \( \delta \) be such that \( |u(z) - u(\xi)| < \varepsilon \) for \( z \) in \( \Delta(\xi, \delta) \). Let \( \Gamma \) be a chord of \( \Delta(\xi, \delta) \) which is parallel to the tangent to \( \Delta(\zeta_0, \rho_0) \) at \( \xi \) and which lies outside \( \Delta(\zeta_0, \rho_0) \). Let \( D \) be the larger part of \( \Delta(\xi, \delta) \) cut off by \( \Gamma \). Given an upper bound \( M \) for \( h_2(z) \) on \( D \), let

\[
H(z) = (M + \varepsilon) \left\{ \frac{\pi - \theta}{\pi - \theta_0} \right\} + (u(\xi) + \varepsilon) \left\{ 1 - \frac{\pi - \theta}{\pi - \theta_0} \right\}, \quad z \in D,
\]

where \( \theta = \theta(z) \) is the angle subtended by \( \Gamma \) at \( z \) and \( \theta_0 \) is the constant value of \( \theta \) on the circular part of \( \partial D \). \( H(z) \) is harmonic in \( D \) with boundary values \( u(\xi) + \varepsilon \) on \( \Gamma \) and \( M + \varepsilon \) on \( \partial D \setminus \Gamma \). It follows that for all large \( n \), \( v_{\zeta_n}(z) \leq H(z) \) in \( D_n = D \cap \Delta(\zeta_n, \rho(\zeta_n)) \). For when \( n \) is large, \( \partial D_n \) consists of a part of \( \partial D \setminus \Gamma \), on which \( H(z) = M + \varepsilon \geq v_{\zeta_n}(z) \), together with the part of the boundary of \( \Delta(\zeta_n, \rho(\zeta_n)) \) that lies in \( D \), and there \( v_{\zeta_n}(z) = u(z) < u(\xi) + \varepsilon \leq H(z) \). Hence
\[ \lim v_{\zeta_n}(z_n) \leq H(\xi) \] and thus, allowing \( D \) to contract to a half-disc, \( \lim v_{\zeta_n}(z_n) \leq u(\xi) + \varepsilon \). A similar argument gives \( \lim v_{\zeta_n}(z_n) \geq u(\xi) - \varepsilon \). Since \( \varepsilon \) is arbitrary (2.4) follows, contradicting the assumption and proving the lemma.

**Proof of (2.2).** \( V(z) \) is upper-semicontinuous. For otherwise there is a point \( \xi \) and a sequence \( z_n \to \xi \) such that \( V(z_n) \to V_0 > V(\xi) \). It follows that there is another sequence \( \zeta_n \) such that \( v_{\zeta_n}(z_n) \to V_0 \), and since \( V(\xi) \geq u(\xi) \) we conclude from the lemma that for some \( \rho_0 > 0, \Delta(\xi, \rho_0) \subseteq \Delta(\zeta_n, \rho(\zeta_n)) \) for all large \( n \). Being uniformly bounded there, \( v_{\zeta_n}(z) \) has a subsequence which converges uniformly in \( \Delta(\xi, \frac{1}{2} \rho_0) \), and therefore (for the subsequence) \( v_{\zeta_n}(\xi) \to V_0 \). Hence \( V(\xi) \geq V_0 \), a contradiction.

To see that \( V(z) \) is subharmonic, fix \( \xi \) and distinguish two cases, the first of which is \( V(\xi) = u(\xi) \). For all positive \( r < \rho(\xi) \)

\[
V(\xi) = u(\xi) = v_{\xi}(\xi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v_{\xi}(\xi + re^{i\theta}) \, d\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} V(\xi + re^{i\theta}) \, d\theta.
\]

In the second case \( V(\xi) > u(\xi) \), and then there is a sequence \( \zeta_n \) such that \( v_{\zeta_n}(\xi) \to V(\xi) \). From the lemma it follows that there must be a neighborhood of \( \xi \) contained in \( \Delta(\zeta_n, \rho(\zeta_n)) \) for all large \( n \), within which a subsequence of \( v_{\zeta_n}(z) \) converges uniformly to some harmonic limit \( g(z) \leq V(z) \). Hence for all small positive \( r \)

\[
V(\xi) = g(\xi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\xi + re^{i\theta}) \, d\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} V(\xi + re^{i\theta}) \, d\theta.
\]

**Proof of (2.3).** Let \( w = e^{i\theta} \) be a point where \( u(z) \) has a limit \( u(w) \). We show that \( \lim V(z) \leq u(w) \); the other inequality \( \lim V(z) \geq u(w) \) is proved similarly.

Given \( \varepsilon > 0 \) let \( \delta \) be such that \( |u(z) - u(w)| < \varepsilon \) for all \( z \) in \( G = \Delta(0,1) \cap \Delta(w, \delta) \). Since \( h_2(z) - h_1(z) \) has finite nontangential limits a.e., and \( u(z) \) is between \( h_1(z) \) and \( h_2(z) \), it may be supposed that \( \delta \) is chosen so that \( h_2(z) \) is bounded (by \( N \) say) on \( \Delta(0,1) \cap \partial G \).

Let \( D \) be the half-disc with center \( w \) and radius \( \delta \) which contains \( G \), and for \( z \) in \( D \) define

\[
J(z) = \frac{2}{\pi} (N + \varepsilon)(\pi - \theta) + \frac{2}{\pi} (u(w) + \varepsilon) \left( \theta - \frac{1}{2\pi} \right),
\]

where \( \theta = \theta(z) \) is the angle subtended at \( z \) by the diameter of \( D \). \( J(z) \) is harmonic in \( D \) with boundary values \( u(w) + \varepsilon \) on the diameter and \( N + \varepsilon \) elsewhere. Now if \( z \) is in \( D \) and \( |z - \zeta| < \rho(\zeta) \), then \( v_{\zeta}(z) \leq J(z) \). The boundary of \( D \cap \Delta(\zeta, \rho(\zeta)) \) consists of at most two circular arcs, one of which is contained in \( D \), the other being part of the semicircumference of \( D \), and on the first of these \( v_{\zeta}(z) = u(z) < u(w) + \varepsilon \leq J(z) \), while on the other \( v_{\zeta}(z) \leq h_2(z) \leq N < J(z) \). Hence \( V(z) \leq J(z) \) in \( D \) which gives \( \lim V(z) \leq u(w) + \varepsilon \) for any \( \varepsilon > 0 \).

**3. Proof of Theorem 1.** \( V(z) - W(z) \) is a nonnegative subharmonic function bounded above by \( h_2(z) - h_1(z) \) and so can be represented as

\[
V(z) - W(z) = H(z) + K(z),
\]

where \( H(z) \) is the least harmonic majorant of \( V(z) - W(z) \) and \( K(z) \) is a nonpositive subharmonic function with radial limits 0 a.e. \([6, p. 172]\). It follows from (2.3) that \( H(z) \) has radial limits 0 a.e., and is, from (ii) of Theorem 1, bounded along every radius. From a uniqueness theorem due to Lohwater \([3]\), \( H(z) \equiv 0 \). Since the
left-hand side of (3.1) is nonnegative while the right-hand side is nonpositive both are zero, i.e. \( V(z) = W(z) = u(z) \). \( u(z) \) is thus harmonic, being subharmonic and superharmonic.

4. Proof of Theorem 2. The first part of the proof borrows from Lohwater’s argument [3]. Let \( \mu(t) \) be the nondecreasing function occurring in the Herglotz representation of \( h \), and let \( \mu(t) = \nu(t) + \sigma(t) \) be its Lebesgue decomposition, where \( \nu(t) \) is absolutely continuous and \( \sigma(t) \) is singular. Lohwater has shown [4] that if \( \mu(t) \) is discontinuous at \( T \) then \( h(re^{it}) \to +\infty \) as \( r \to 1 \). It follows that if (1.2) holds for every \( \theta \) then \( \sigma(t) \) is continuous and so, according to an observation by Saks [5, p. 128], either \( \sigma'(t) = +\infty \) on an uncountable set or else \( \sigma(t) \) is constant. Under the first alternative \( h(z) \) has radial limits \( +\infty \) on an uncountable set, which is contrary to (1.2). Thus \( \sigma(t) \) is constant, which gives (1.3) with \( H(t) = \nu'(t) \).

Assume now that (1.4) fails for some \( \theta \), which may be taken to be 0. There is then a sequence \( \delta_n \to 0 \) such that

\[
\int_{-\delta_n}^{\delta_n} H(t) \, dt \geq 2n\delta_n,
\]

and therefore, for \( \delta_n \leq t \leq 2\delta_n \),

\[
\int_{-t}^{t} H(t) \, dt \geq nt.
\]

Integration by parts gives

\[
(4.1) \quad h(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(t) \, dt \cdot \frac{1-r}{1+r} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \int_{0}^{t} H(s) \, ds \right) \frac{2r(1-r^2)\sin t}{(1-2r\cos t + r^2)^2} \, dt,
\]

and the second term on the right is, for all large \( n \),

\[
\frac{1}{2\pi} \int_{0}^{\pi} \left( \int_{-t}^{t} H(s) \, ds \right) \frac{2r(1-r^2)\sin t}{(1-2r\cos t + r^2)^2} \, dt \geq \frac{1}{\pi} \frac{r(1-r^2)n\delta_n^2}{(1-r^2)^2 + 4r\sin^2 \frac{1}{2} t} \, dt \geq \frac{1}{\pi} \frac{r(1-r^2)n\delta_n^2}{(1-r^2)^2 + 4r\delta_n^2} \cdot \frac{1}{2\pi} \int_{0}^{\pi} \left( \int_{-t}^{t} H(s) \, ds \right) \frac{2r(1-r^2)\sin t}{(1-2r\cos t + r^2)^2} \, dt.
\]

Let \( r_n \) be the solution of \( (1-r_n)r_n^{-1/2} = 2\delta_n \) which approaches 1 as \( n \to \infty \). Then, from (4.1), \( h(r_n) \geq \frac{1}{16} \pi^{-2} n r_n^{-1/2} (1 + r_n) + o(1) \to +\infty \) as \( n \to \infty \), contradicting (1.2).

Conversely, suppose that \( h(z) \) is given by (1.3) and that (1.4) holds for every \( \theta \). Since \( t^{-1} \int_{0}^{t} H(s) \, ds \) is bounded for all \( t \in (-\pi, \pi) \), we obtain from (4.1)

\[
h(r) \leq o(1) + O \left\{ (1-r^2) \int_{-\pi}^{\pi} \frac{2rt\sin t}{(1-2r\cos t + r^2)^2} \, dt \right\} = O(1),
\]

as \( r \to 1^- \). By a rotation the result for any \( \theta \) follows.
5. $h^1$ functions. It remains to justify the remark following Theorem 2. For $n \geq 2$, let

$$H_n(t) = \begin{cases} 
  n^3, & n^{-1} - n^{-3} < t < n^{-1}, \\
  -n^3, & n^{-1} < t < n^{-1} + n^{-3}, \\
  0, & \text{elsewhere},
\end{cases}$$

and let $h_n(z)$ be given by (1.3) with $H$ replaced by $H_n$. After slight simplification we obtain

$$0 \leq h_n(r) = \frac{2r}{\pi} (1 - r^2)n^3 \sin n^{-1}$$

$$\times \int_0^{n^{-3}} \frac{\sin t}{[1 - 2r \cos (n^{-1} + t) + r^2][1 - 2r \cos (n^{-1} - t) + r^2]} \, dt$$

$$\leq (1 - r^2)n^{-4}[1 - 2r \cos (n^{-1} - n^{-3}) + r^2]^{-2}$$

$$\leq (1 - r^2)n^{-4} \sin (n^{-1} - n^{-3})^{-4} \leq K(1 - r^2)$$

for some constant $K$. Also $\int_{-T}^{T} H_n(t) \, dt = nT$ when $T = n_1$ and $0$ for $n^{-1} + n^{-3} \leq T \leq \pi$ and $0 \leq T \leq n^{-1} - n^{-3}$.

Now define

$$h(z) = \sum_{j=1}^{\infty} 2^{-j} h_{n_j}(z), \quad H(t) = \sum_{j=1}^{\infty} 2^{-j} H_{n_j}(z),$$

where $n_j = j2^j$.

Certainly $h(z) \in h^1$. Also, since the intervals in (5.1) for various $n$ do not overlap, we have (1.2) for $\theta \neq 0$ and, in view of (5.2), for $\theta = 0$ also. Finally with $T = n_1^{-1}$,

$$\int_{-T}^{T} H(t) \, dt = \int_{-T}^{T} 2^{-j} H_{n_j}(t) \, dt = 2^{-j} n_j T = jT,$$

so (1.4) fails at $\theta = 0$.

REFERENCES


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