PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 100, Number 3, July 1987

ON THE RESTRICTED MEAN VALUE PROPERTY

P. C. FENTON

ABSTRACT. Suppose that u is continuous in the open unit disc and has the restricted mean value property. It is shown that if u has finite boundary limits almost everywhere, and if u possesses a harmonic majorant and minorant, the difference between which has finite radial upper limits everywhere, then u is harmonic.

1. Introduction. A function $u(z)$ in the open unit disc $\Delta(0,1)$ has the *restricted mean-value* (rmv) property if for each $z \in \Delta(0,1)$ there is a positive number $\rho = \rho(z) < 1 - |z|$ such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(z+\rho e^{i\theta})\,d\theta = u(z).$$

Here we shall prove

THEOREM 1. *Suppose that $u(z)$ is continuous in $\Delta(0,1)$ and has the rmv property. If*

(i) $\lim_{z\to e^{i\theta}} u(z)$ *exists for almost all θ, and*

(ii) $u(z)$ *has a harmonic majorant $h_2(z)$ and a harmonic minorant $h_1(z)$ such that for all $\theta \in [-\pi,\pi]$*

$$\varlimsup_{r\to 1}\{h_2(re^{i\theta}) - h_1(re^{i\theta})\} < \infty, \tag{1.1}$$

then u is harmonic.

This is an extension of an earlier result of the author [**1**] in which instead of condition (ii) it is supposed the $u(z)$ is bounded. Both are motivated by problem 11 in J. E. Littlewood's book [**2**]. It is a genuine extension since $h_2(z) - h_1(z)$ need not be bounded, as the following theorem shows.

THEOREM 2. *Suppose that $h(z)$ is a nonnegative harmonic function in $\Delta(0,1)$ such that*

$$\varlimsup_{r\to 1} h(re^{i\theta}) < \infty \tag{1.2}$$

for every $\theta \in [-\pi,\pi]$. Then

$$h(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(t)re\{(e^{it}+z)/(e^{it}-z)\}\,dt, \tag{1.3}$$

where $H(t)$ is nonnegative and such that

$$\varlimsup_{T\downarrow 0}\frac{1}{2T}\int_{-T}^{T} H(t+\theta)\,dt < \infty \tag{1.4}$$

Received by the editors May 1, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 31A05.

for every $\theta \in [-\pi, \pi]$. Conversely, if $h(z)$ is defined by (1.3), *where $H(t)$ is nonnegative and satisfies* (1.4) *for every $\theta \in [-\pi, \pi]$, then* (1.2) *holds for every $\theta \in [-\pi, \pi]$.*

As will be shown later, Theorem 2 has no natural counterpart for h^1 functions at least in the sense that there is an h^1 function $h(z)$ of the form (1.3) satisfying $\overline{\lim}_{r\to 1} |h(re^{i\theta})| < \infty$ for every $\theta \in [-\pi, \pi]$ for which

$$\overline{\lim_{T\downarrow 0}} \left| \frac{1}{2T} \int_{-T}^{T} H(t)\, dt \right| = +\infty.$$

2. Two auxiliary functions. Given $\varsigma \in \Delta(0,1)$ let $v_\varsigma(z)$ be the harmonic function in $\Delta(\varsigma, \rho(\varsigma))$ which is the Poisson integral of the boundary values of u. Define

$$V(z) = \sup v_\varsigma(z), \quad W(z) = \inf v_\varsigma(z), \tag{2.1}$$

the supremum and infimum extended over those ς such that $|z - \varsigma| < \rho(\varsigma)$. From condition (ii) both $V(z)$ and $W(z)$ are finite for all z and

$$h_2(z) \geq V(z) \geq v_z(z) = u(z) \geq W(z) \geq h_1(z).$$

These V and W are slightly simpler versions of functions introduced in [**1**].

The significant properties of V and W are

(2.2) *$V(z)$ is subharmonic and $W(z)$ is superharmonic,*

and

(2.3) *for a.a. $\theta \in [-\pi, \pi]$,* $\displaystyle\lim_{z\to e^{i\theta}} V(z) = \lim_{z\to e^{i\theta}} W(z) = \lim_{z\to e^{i\theta}} u(z).$

To prove these we need

LEMMA. *Let $\xi \in \Delta(0,1)$ and let $z_n \to \xi$. If ς_n is such that $|z_n - \varsigma_n| < \rho(\varsigma_n)$ for all n and if $\rho(\varsigma_n) - |z_n - \varsigma_n| \to 0$ as $n \to \infty$, then*

$$v_{\varsigma_n}(z_n) \to u(\xi). \tag{2.4}$$

Suppose that (2.4) is false for some sequence ς_n (which may be assumed without loss to be convergent) with limit ς_0 say. Evidently $\rho_0 \neq 0$, where $\rho_0 = \lim \rho(\varsigma_n)$, so that ξ is a boundary point of $\Delta(\varsigma_0, \rho_0)$.

Given $\varepsilon > 0$ let δ be such that $|u(z) - u(\xi)| < \varepsilon$ for z in $\Delta(\xi, \delta)$. Let Γ be a chord of $\Delta(\xi, \delta)$ which is parallel to the tangent to $\Delta(\varsigma_0, \rho_0)$ at ξ and which lies outside $\Delta(\varsigma_0, \rho_0)$. Let D be the larger part of $\Delta(\xi, \delta)$ cut off by Γ. Given an upper bound M for $h_2(z)$ on D, let

$$H(z) = (M + \varepsilon) \left\{ \frac{\pi - \theta}{\pi - \theta_0} \right\} + (u(\xi) + \varepsilon) \left\{ 1 - \frac{\pi - \theta}{\pi - \theta_0} \right\}, \qquad z \in D,$$

where $\theta = \theta(z)$ is the angle subtended by Γ at z and θ_0 is the constant value of θ on the circular part of ∂D. $H(z)$ is harmonic in D with boundary values $u(\xi) + \varepsilon$ on Γ and $M + \varepsilon$ on $\partial D \backslash \Gamma$. It follows that for all large n, $v_{\varsigma_n}(z) \leq H(z)$ in $D_n = D \cap \Delta(\varsigma_n, \rho(\varsigma_n))$. For when n is large, ∂D_n consists of a part of $\partial D \backslash \Gamma$, on which $H(z) = M + \varepsilon \geq v_{\varsigma_n}(z)$, together with the part of the boundary of $\Delta(\varsigma_n, \rho(\varsigma_n))$ that lies in D, and there $v_{\varsigma_n}(z) = u(z) < u(\xi) + \varepsilon \leq H(z)$. Hence

$\overline{\lim}\, v_{\varsigma_n}(z_n) \leq H(\xi)$ and thus, allowing D to contract to a half-disc, $\overline{\lim}\, v_{\varsigma_n}(z_n) \leq u(\xi)+\varepsilon$. A similar argument gives $\underline{\lim}\, v_{\varsigma_n}(z_n) \geq u(\xi)-\varepsilon$. Since ε is arbitrary (2.4) follows, contradicting the assumption and proving the lemma.

PROOF OF (2.2). $V(z)$ is upper-semicontinuous. For otherwise there is a point ξ and a sequence $z_n \to \xi$ such that $V(z_n) \to V_0 > V(\xi)$. It follows that there is another sequence ς_n such that $v_{\varsigma_n}(z_n) \to V_0$, and since $V(\xi) \geq u(\xi)$ we conclude from the lemma that for some $\rho_0 > 0$, $\Delta(\xi, \rho_0) \subseteq \Delta(\varsigma_n, \rho(\varsigma_n))$ for all large n. Being uniformly bounded there, $v_{\varsigma_n}(z)$ has a subsequence which converges uniformly in $\Delta(\xi, \frac{1}{2}\rho_0)$, and therefore (for the subsequence) $v_{\varsigma_n}(\xi) \to V_0$. Hence $V(\xi) \geq V_0$, a contradiction.

To see that $V(z)$ is subharmonic, fix ξ and distinguish two cases, the first of which is $V(\xi) = u(\xi)$. For all positive $r < \rho(\xi)$

$$V(\xi) = u(\xi) = v_\xi(\xi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v_\xi(\xi + re^{i\theta})\, d\theta \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} V(\xi + re^{i\theta})\, d\theta.$$

In the second case $V(\xi) > u(\xi)$, and then there is a sequence ς_n such that $v_{\varsigma_n}(\xi) \to V(\xi)$. From the lemma it follows that there must be a neighborhood of ξ contained in $\Delta(\varsigma_n, \rho(\varsigma_n))$ for all large n, within which a subsequence of $v_{\varsigma_n}(z)$ converges uniformly to some harmonic limit $g(z) \leq V(z)$. Hence for all small positive r

$$V(\xi) = g(\xi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\xi + re^{i\theta})\, d\theta \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} V(\xi + re^{i\theta})\, d\theta.$$

PROOF OF (2.3). Let $w = e^{i\theta}$ be a point where $u(z)$ has a limit $u(w)$. We show that $\overline{\lim}\, V(z) \leq u(w)$; the other inequality $\underline{\lim}\, V(z) \geq u(w)$ is proved similarly.

Given $\varepsilon > 0$ let δ be such that $|u(z)-u(w)| < \varepsilon$ for all z in $G = \Delta(0,1) \cap \Delta(w, \delta)$. Since $h_2(z) - h_1(z)$ has finite nontangential limits a.e., and $u(z)$ is between $h_1(z)$ and $h_2(z)$, it may be supposed that δ is chosen so that $h_2(z)$ is bounded (by N say) on $\Delta(0,1) \cap \partial G$.

Let D be the half-disc with center w and radius δ which contains G, and for z in D define

$$J(z) = \frac{2}{\pi}(N+\varepsilon)(\pi - \theta) + \frac{2}{\pi}(u(w)+\varepsilon)\left(\theta - \frac{1}{2}\pi\right),$$

where $\theta = \theta(z)$ is the angle subtended at z by the diameter of D. $J(z)$ is harmonic in D with boundary values $u(w)+\varepsilon$ on the diameter and $N+\varepsilon$ elsewhere. Now if z is in D and $|z-\varsigma| < \rho(\varsigma)$, then $v_\varsigma(z) \leq J(z)$. The boundary of $D \cap \Delta(\varsigma, \rho(\varsigma))$ consists of at most two circular arcs, one of which is contained in D, the other being part of the semicircumference of D, and on the first of these $v_\varsigma(z) = u(z) < u(w)+\varepsilon \leq J(z)$, while on the other $v_\varsigma(z) \leq h_2(z) \leq N < J(z)$. Hence $V(z) \leq J(z)$ in D which gives $\overline{\lim}\, V(z) \leq u(w) + \varepsilon$ for any $\varepsilon > 0$.

3. Proof of Theorem 1. $V(z) - W(z)$ is a nonnegative subharmonic function bounded above by $h_2(z) - h_1(z)$ and so can be represented as

$$V(z) - W(z) = H(z) + K(z), \tag{3.1}$$

where $H(z)$ is the least harmonic majorant of $V(z)-W(z)$ and $K(z)$ is a nonpositive subharmonic function with radial limits 0 a.e. [**6**, p. 172]. It follows from (2.3) that $H(z)$ has radial limits 0 a.e., and is, from (ii) of Theorem 1, bounded along every radius. From a uniqueness theorem due to Lohwater [**3**], $H(z) \equiv 0$. Since the left-hand side of (3.1) is nonnegative while the right-hand side is nonpositive both are zero, i.e. $V(z) = W(z) = u(z)$. $u(z)$ is thus harmonic, being subharmonic and superharmonic.

4. Proof of Theorem 2. The first part of the proof borrows from Lohwater's argument [**3**]. Let $\mu(t)$ be the nondecreasing function occurring in the Herglotz representation of h, and let $\mu(t) = \nu(t) + \sigma(t)$ be its Lebesgue decomposition, where $\nu(t)$ is absolutely continuous and $\sigma(t)$ is singular. Lohwater has shown [**4**] that if $\mu(t)$ is discontinuous at T then $h(re^{iT}) \to +\infty$ as $r \to 1$. It follows that if (1.2) holds for every θ then $\sigma(t)$ is continuous and so, according to an observation by Saks [**5**, p. 128], either $\sigma'(t) = +\infty$ on an uncountable set or else $\sigma(t)$ is constant. Under the first alternative $h(z)$ has radial limits $+\infty$ on an uncountable set, which is contrary to (1.2). Thus $\sigma(t)$ is constant, which gives (1.3) with $H(t) = \nu'(t)$.

Assume now that (1.4) fails for some θ, which may be taken to be 0. There is then a sequence $\delta_n \to 0$ such that

$$\int_{-\delta_n}^{\delta_n} H(t)\,dt \geq 2n\delta_n,$$

and therefore, for $\delta_n \leq t \leq 2\delta_n$,

$$\int_{-t}^{t} H(t)\,dt \geq nt.$$

Integration by parts gives

$$h(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(t)\,dt \cdot \frac{1-r}{1+r} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\int_0^t H(s)\,ds\right\}\frac{2r(1-r^2)\sin t}{(1-2r\cos t+r^2)^2}\,dt, \tag{4.1}$$

and the second term on the right is, for all large n,

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{\pi}\left\{\int_{-t}^{t} H(s)\,ds\right\}\frac{2r(1-r^2)\sin t}{(1-2r\cos t+r^2)^2}\,dt \\
&\quad\geq \frac{1}{\pi} r(1-r^2)n\int_{\delta_n}^{2\delta_n}\frac{t\sin t}{\{(1-r)^2+4r\sin^2\frac{1}{2}t\}^2}\,dt \\
&\quad\geq \frac{1}{\pi} r(1-r^2)n\delta_n^2\sin\delta_n\{(1-r)^2+4r\sin^2\delta_n\}^{-2} \\
&\quad\geq 2\pi^{-2}r(1-r^2)n\delta_n^3\{(1-r)^2+4r\delta_n^2\}^{-2}.
\end{aligned}$$

Let r_n be the solution of $(1-r_n)r_n^{-1/2} = 2\delta_n$ which approaches $1-$ as $n \to \infty$. Then, from (4.1), $h(r_n) \geq \frac{1}{16}\pi^{-2}nr_n^{-1/2}(1+r_n) + o(1) \to +\infty$ as $n \to \infty$, contradicting (1.2).

Conversely, suppose that $h(z)$ is given by (1.3) and that (1.4) holds for every θ. Since $t^{-1}\int_0^t H(s)\,ds$ is bounded for all $t \in (-\pi, \pi)$, we obtain from (4.1)

$$h(r) \leq o(1) + O\left\{(1-r^2)\int_{-\pi}^{\pi}\frac{2rt\sin t}{(1-2r\cos t+r^2)^2}dt\right\} = O(1),$$

as $r \to 1-$. By a rotation the result for any θ follows.

5. h^1 functions. It remains to justify the remark following Theorem 2. For $n \geq 2$, let

$$H_n(t) = \begin{cases} n^3, & n^{-1} - n^{-3} < t < n^{-1}, \\ -n^3, & n^{-1} < t < n^{-1} + n^{-3}, \\ 0, & \text{elsewhere}, \end{cases} \tag{5.1}$$

and let $h_n(z)$ be given by (1.3) with H replaced by H_n. After slight simplification we obtain

$$\begin{aligned} 0 \leq h_n(r) &= \frac{2r}{\pi}(1-r^2)n^3 \sin n^{-1} \\ &\times \int_0^{n^{-3}} \frac{\sin t}{[1-2r\cos(n^{-1}+t)+r^2][1-2r\cos(n^{-1}-t)+r^2]}\,dt \\ &\leq (1-r^2)n^{-4}[1-2r\cos(n^{-1}-n^{-3})+r^2]^{-2} \\ &\leq (1-r^2)n^{-4}[\sin(n^{-1}-n^{-3})]^{-4} \leq K(1-r^2) \end{aligned} \tag{5.2}$$

for some constant K. Also $\int_{-T}^{T} H_n(t)\, dt = nT$ when $T = n_1$ and $= 0$ for $n^{-1} + n^{-3} \leq T \leq \pi$ and $0 \leq T \leq n^{-1} - n^{-3}$.

Now define

$$h(z) = \sum_{j=1}^{\infty} 2^{-j} h_{n_j}(z), \qquad H(t) = \sum_{j=1}^{\infty} 2^{-j} H_{n_j}(z),$$

where $n_j = j2^j$.

Certainly $h(z) \in h^1$. Also, since the intervals in (5.1) for various n do not overlap, we have (1.2) for $\theta \neq 0$ and, in view of (5.2), for $\theta = 0$ also. Finally with $T = n_j^{-1}$,

$$\int_{-T}^{T} H(t)\, dt = \int_{-T}^{T} 2^{-j} H_{n_j}(t)\, dt = 2^{-j} n_j T = jT,$$

so (1.4) fails at $\theta = 0$.

References

1. P. C. Fenton, *Functions having the restricted mean value property*, J. London Math. Soc. (2) **14** (1976), 451–458.
2. J. E. Littlewood, *Some problems in real and complex analysis*, Heath, Boston, Mass., 1968.
3. A. J. Lohwater, *A uniqueness theorem for a class of harmonic functions*, Proc. Amer. Math. Soc. **3** (1952), 278–279.
4. ———, *The boundary values of a class of meromorphic functions*, Duke Math. J. **19** (1952), 243–252.
5. S. Saks, *Theory of the integral*, PWN, Warsaw, 1937.
6. M. Tsuji, *Potential theory in modern function theory*, Chelsea, New York.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OTAGO, POB 56, DUNEDIN, NEW ZEALAND