MINIMAL PERIODIC ORBITS AND TOPOLOGICAL ENTROPY
OF INTERVAL MAPS

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ABSTRACT. For any two integers \( m \geq 0 \) and \( n \geq 1 \), we construct continuous functions from \([0, 1]\) into itself which have exactly one minimal periodic orbit of least period \( 2^m(2n + 1) \), but with topological entropy equal to \( \infty \).

Introduction. Let \( I \) denote the unit interval \([0, 1]\) and let \( g \in C^0(I, I) \). If \( g \) has a periodic point of least period \( 2^m(2n + 1) \), where \( m \geq 0 \) and \( n \geq 1 \) are integers, then it is well known [3] that the topological entropy of \( g \) is greater than or equal to \( (\log \lambda_n)/2^m \), where \( \lambda_n \) is the (unique) positive zero of the polynomial \( x^{2n+1} - 2x^{2n-1} - 1 \). The converse is false, but known counterexamples are rather complicated [4, p. 407]. The purpose of this note is to indicate how to use an easy and well-known method to construct examples which are simpler but with stronger properties than those given in [4, p. 407] except that our examples are not piecewise monotone.

As a consequence of our construction, we also obtain the well-known example \( g_\infty \) as described in [8, p. 14] which has exactly one periodic orbit of least period \( 2^m \) for every \( m \geq 0 \) and no other periodic orbits. It is worth mentioning that the set of all periodic points of the example \( g_\infty \) described in [8, p. 14] is not closed. This is in contrast to the fact [7] that if the set of all periodic points of a continuous function in \( C^0(I, I) \) is closed, then this function can only have periodic points of periods some powers of 2.

The construction. For every continuous function \( g \) in \( C^0(I, I) \), let \( G : [0, 3] \to [0, 3] \) be the continuous function defined by (i) \( G(x) = g(x) + 2 \) for \( 0 < x < 1 \); (ii) \( G(x) = x - 2 \) for \( 2 < x < 3 \); and (iii) \( G \) is linear on \([1, 2]\). Then it is clear that \( G^2|I = g \). Now let \( \tilde{g} \) be the scaled-down copy of \( G \) on \( I \). That is, \( \tilde{g}(x) = [g(3x)+2]/3 \) for \( 0 < x < 1/3 \); \( \tilde{g}(x) = [2 + g(1)](2/3 - x) \) for \( 1/3 \leq x < 2/3 \); and \( \tilde{g}(x) = x - 2/3 \) for \( 2/3 < x < 1 \). It follows from [1] that the topological entropy of \( \tilde{g} \) is greater than or equal to one half of that of \( g \). This function \( \tilde{g} \) is called the renormalized square root of \( g \) on \( I \). For every continuous function \( g_0 \) in \( C^0(I, I) \), we define the sequence \( \langle g_m \rangle \) (\( m \geq 1 \)) inductively by letting \( g_m \) be the renormalized square root of \( g_{m-1} \) on \( I \). This sequence \( \langle g_m \rangle \) (\( m \geq 1 \)) is called the sequence of successive renormalized square roots of \( g_0 \) on \( I \).

For every positive integer \( k \), choose \( 2k + 2 \) real numbers \( a_{k,i} \) with \( 0 = a_{k,0} < a_{k,1} < a_{k,2} < \cdots < a_{k,2k+1} = 1 \). Let \( p_k \) be the continuous function in \( C^0(I, I) \) defined by (i) \( p_k(a_{k,i}) = 0 \) for all even \( i \); (ii) \( p_k(a_{k,i}) = 1 \) for all odd \( i \); and (iii) \( p_k \) is linear on each interval \([a_{k,i}, a_{k,i+1}]\), \( 0 \leq i \leq 2k \). Let \( q_k \) be the continuous function...
from the interval \([1/(k+1), 1/k]\) onto itself which is the scaled-down copy of \(p_k\) on \([1/(k+1), 1/k]\). That is,
\[
q_k(x) = 1/(k+1) + p_k(k(k+1)(x - 1/(k+1)))/[k(k+1)].
\]
Finally, let \(f_0 \in C^0(I, I)\) be the continuous function defined by \(f_0(0) = 0\) and 
f_0(x) = q_k(x) on \([1/(k+1), 1/k]\) for each positive integer \(k\) and let \(\langle f_m \rangle (m \geq 1)\) be the sequence of successive renormalized square roots of \(f_0\) on \(I\). Now we can state the following theorem whose proof is easy and omitted. (For the definition of minimal periodic orbits, see [2 or 5].)

**Theorem 1.** Let the sequence \(\langle f_m \rangle (m \geq 0)\) be defined as above. Then \(\langle f_m \rangle\) is a uniformly convergent sequence in \(C^0(I, I)\) with the following two properties:

1. For every integer \(m \geq 0\), \(f_m\) has infinitely many minimal periodic orbits of least period \(2^m \cdot 3\) and the topological entropy of \(f_m\) is \(\infty\).
2. If \(f\) is the uniform limit of the sequence \(\langle f_m \rangle\), then \(f\) is exactly the same as the function \(g_\infty\) described in [8, p. 14] with zero topological entropy [6].

In the above theorem, every function \(f_k\) has minimal periodic orbits of least period \(2^k \cdot 3\). In the following, we will use these functions \(f_k\) to construct, for any two integers \(m \geq 0\) and \(n \geq 1\), continuous functions \(F_{m,n}\) in \(C^0(I, I)\) which have exactly one minimal periodic orbit of least period \(2^m(2n + 1)\), but with topological entropy equal to \(\infty\).

For every positive integer \(n\), let \(u_n\) be any continuous function from \([2/3, 1]\) into itself with exactly one minimal periodic orbit [9] (see [2, 5] also) of least period \(2n + 1\) and let \(F_{0,n}\) be the continuous function in \(C^0(I, I)\) defined by (i) \(F_{0,n}(x) = f_k(3x)/3\) for \(0 \leq x \leq 1/3\), where \(k\) is any positive integer and \(f_k\) is defined as in Theorem 1; (ii) \(F_{0,n}(x) = u_n(x)\) for \(2/3 \leq x \leq 1\); and (iii) \(F_{0,n}\) is linear on \([1/3, 2/3]\). It is clear that \(F_{0,n}\) has exactly one minimal periodic orbit of least period \(2n + 1\) and its topological entropy is \(\infty\). For any fixed integer \(n > 0\), let \(\langle F_{m,n} \rangle (m \geq 1)\) be the sequence of successive renormalized square roots of \(F_{0,n}\) on \(I\).

Now we can state the following theorem whose proof is again easy and omitted.

**Theorem 2.** For every positive integer \(n\), let the sequence \(\langle F_{m,n} \rangle (m \geq 0)\) be defined as above. Then for every fixed \(n > 0\), \(\langle F_{m,n} \rangle (m \geq 0)\) is a uniformly convergent sequence in \(C^0(I, I)\) with the following two properties:

1. For every integer \(m \geq 0\), the function \(F_{m,n}\) has exactly one minimal periodic orbit of least period \(2^m(2n + 1)\) and the topological entropy of \(F_{m,n}\) is \(\infty\).
2. If \(F_n\) is the uniform limit of the sequence \(\langle F_{m,n} \rangle\), then \(F_n = f\), where \(f\) is defined as in Theorem 1.

**Remark.** We can also construct functions \(G_{m,n}\) in \(C^0(I, I)\) with the properties as stated in part (1) of Theorem 2 as follows: For any two integers \(m \geq 0\) and \(n \geq 1\), let \(v_{m,n}\) be any continuous function from \([2/3, 1]\) into itself which has exactly one minimal periodic orbit [2, 5] of least period \(2^m(2n + 1)\). Let \(G_{m,n}\) be the continuous function in \(C^0(I, I)\) defined by (i) \(G_{m,n}(x) = f_k(3x)/3\) for \(0 \leq x \leq 1/3\), where \(k > m\) is any integer and \(f_k\) is defined as in Theorem 1; (ii) \(G_{m,n}(x) = v_{m,n}(x)\) for \(2/3 \leq x \leq 1\); and (iii) \(G_{m,n}\) is linear on \([1/3, 2/3]\). Then it is easy to see that \(G_{m,n}\) has exactly one minimal periodic orbit of least period \(2^m(2n + 1)\) and the topological entropy of \(G_{m,n}\) is \(\infty\).
REFERENCES

5. C.-W. Ho, On the structure of the minimum orbits of periodic points for maps of the real line (to appear).

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