

MINIMAL PERIODIC ORBITS AND TOPOLOGICAL ENTROPY OF INTERVAL MAPS

BAU-SEN DU

ABSTRACT. For any two integers $m \geq 0$ and $n \geq 1$, we construct continuous functions from $[0, 1]$ into itself which have exactly one minimal periodic orbit of least period $2^m(2n + 1)$, but with topological entropy equal to ∞ .

Introduction. Let I denote the unit interval $[0, 1]$ and let $g \in C^0(I, I)$. If g has a periodic point of least period $2^m(2n + 1)$, where $m \geq 0$ and $n \geq 1$ are integers, then it is well known [3] that the topological entropy of g is greater than or equal to $(\log \lambda_n)/2^m$, where λ_n is the (unique) positive zero of the polynomial $x^{2n+1} - 2x^{2n-1} - 1$. The converse is false, but known counterexamples are rather complicated [4, p. 407]. The purpose of this note is to indicate how to use an easy and well-known method to construct examples which are simpler but with stronger properties than those given in [4, p. 407] except that our examples are not piecewise monotone. As a consequence of our construction, we also obtain the well-known example g_∞ as described in [8, p. 14] which has exactly one periodic orbit of least period 2^m for every $m \geq 0$ and no other periodic orbits. It is worth mentioning that the set of all periodic points of the example g_∞ described in [8, p. 14] is not closed. This is in contrast to the fact [7] that if the set of all periodic points of a continuous function in $C^0(I, I)$ is closed, then this function can only have periodic points of periods some powers of 2.

The construction. For every continuous function g in $C^0(I, I)$, let $G: [0, 3] \rightarrow [0, 3]$ be the continuous function defined by (i) $G(x) = g(x) + 2$ for $0 \leq x \leq 1$; (ii) $G(x) = x - 2$ for $2 \leq x \leq 3$; and (iii) G is linear on $[1, 2]$. Then it is clear that $G^2|I = g$. Now let \tilde{g} be the scaled-down copy of G on I . That is, $\tilde{g}(x) = [g(3x) + 2]/3$ for $0 \leq x \leq 1/3$; $\tilde{g}(x) = [2 + g(1)](2/3 - x)$ for $1/3 \leq x \leq 2/3$; and $\tilde{g}(x) = x - 2/3$ for $2/3 \leq x \leq 1$. It follows from [1] that the topological entropy of \tilde{g} is greater than or equal to one half of that of g . This function \tilde{g} is called the renormalized square root of g on I . For every continuous function g_0 in $C^0(I, I)$, we define the sequence $\langle g_m \rangle$ ($m \geq 1$) inductively by letting g_m be the renormalized square root of g_{m-1} on I . This sequence $\langle g_m \rangle$ ($m \geq 1$) is called the sequence of successive renormalized square roots of g_0 on I .

For every positive integer k , choose $2k + 2$ real numbers $a_{k,i}$ with $0 = a_{k,0} < a_{k,1} < a_{k,2} < \dots < a_{k,2k+1} = 1$. Let p_k be the continuous function in $C^0(I, I)$ defined by (i) $p_k(a_{k,i}) = 0$ for all even i ; (ii) $p_k(a_{k,i}) = 1$ for all odd i ; and (iii) p_k is linear on each interval $[a_{k,i}, a_{k,i+1}]$, $0 \leq i \leq 2k$. Let q_k be the continuous function

Received by the editors April 20, 1985.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 26A18; Secondary 54C70, 58F20.

Key words and phrases. Least period, periodic points, periodic orbits, minimal periodic orbits, topological entropy.

from the interval $[1/(k + 1), 1/k]$ onto itself which is the scaled-down copy of p_k on $[1/(k + 1), 1/k]$. That is,

$$q_k(x) = 1/(k + 1) + p_k(k(k + 1)(x - 1/(k + 1)))/[k(k + 1)].$$

Finally, let $f_0 \in C^0(I, I)$ be the continuous function defined by $f_0(0) = 0$ and $f_0(x) = q_k(x)$ on $[1/(k + 1), 1/k]$ for each positive integer k and let $\langle f_m \rangle$ ($m \geq 1$) be the sequence of successive renormalized square roots of f_0 on I . Now we can state the following theorem whose proof is easy and omitted. (For the definition of minimal periodic orbits, see [2 or 5].)

THEOREM 1. *Let the sequence $\langle f_m \rangle$ ($m \geq 0$) be defined as above. Then $\langle f_m \rangle$ is a uniformly convergent sequence in $C^0(I, I)$ with the following two properties:*

(1) *For every integer $m \geq 0$, f_m has infinitely many minimal periodic orbits of least period $2^m \cdot 3$ and the topological entropy of f_m is ∞ .*

(2) *If f is the uniform limit of the sequence $\langle f_m \rangle$, then f is exactly the same as the function g_∞ described in [8, p. 14] with zero topological entropy [6].*

In the above theorem, every function f_k has minimal periodic orbits of least period $2^k \cdot 3$. In the following, we will use these functions f_k to construct, for any two integers $m \geq 0$ and $n \geq 1$, continuous functions $F_{m,n}$ in $C^0(I, I)$ which have exactly one minimal periodic orbit of least period $2^m(2n + 1)$, but with topological entropy equal to ∞ .

For every positive integer n , let u_n be any continuous function from $[2/3, 1]$ into itself with exactly one minimal periodic orbit [9] (see [2, 5] also) of least period $2n + 1$ and let $F_{0,n}$ be the continuous function in $C^0(I, I)$ defined by (i) $F_{0,n}(x) = f_k(3x)/3$ for $0 \leq x \leq 1/3$, where k is any positive integer and f_k is defined as in Theorem 1; (ii) $F_{0,n}(x) = u_n(x)$ for $2/3 \leq x \leq 1$; and (iii) $F_{0,n}$ is linear on $[1/3, 2/3]$. It is clear that $F_{0,n}$ has exactly one minimal periodic orbit of least period $2n + 1$ and its topological entropy is ∞ . For any fixed integer $n > 0$, let $\langle F_{m,n} \rangle$ ($m \geq 1$) be the sequence of successive renormalized square roots of $F_{0,n}$ on I .

Now we can state the following theorem whose proof is again easy and omitted.

THEOREM 2. *For every positive integer n , let the sequence $\langle F_{m,n} \rangle$ ($m \geq 0$) be defined as above. Then for every fixed $n > 0$, $\langle F_{m,n} \rangle$ ($m \geq 0$) is a uniformly convergent sequence in $C^0(I, I)$ with the following two properties:*

(1) *For every integer $m \geq 0$, the function $F_{m,n}$ has exactly one minimal periodic orbit of least period $2^m(2n + 1)$ and the topological entropy of $F_{m,n}$ is ∞ .*

(2) *If F_n is the uniform limit of the sequence $\langle F_{m,n} \rangle$, then $F_n = f$, where f is defined as in Theorem 1.*

REMARK. We can also construct functions $G_{m,n}$ in $C^0(I, I)$ with the properties as stated in part (1) of Theorem 2 as follows: For any two integers $m \geq 0$ and $n \geq 1$, let $v_{m,n}$ be any continuous function from $[2/3, 1]$ into itself which has exactly one minimal periodic orbit [2, 5] of least period $2^m(2n + 1)$. Let $G_{m,n}$ be the continuous function in $C^0(I, I)$ defined by (i) $G_{m,n}(x) = f_k(3x)/3$ for $0 \leq x \leq 1/3$, where $k > m$ is any integer and f_k is defined as in Theorem 1; (ii) $G_{m,n}(x) = v_{m,n}(x)$ for $2/3 \leq x \leq 1$; and (iii) $G_{m,n}$ is linear on $[1/3, 2/3]$. Then it is easy to see that $G_{m,n}$ has exactly one minimal periodic orbit of least period $2^m(2n + 1)$ and the topological entropy of $G_{m,n}$ is ∞ .

REFERENCES

1. R. Adler, A. Konheim, and M. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965), 309–319.
2. L. Alsedá, J. Llibre, and R. Serra, *Minimal periodic orbits for continuous maps of the interval*, Trans. Amer. Math. Soc. **286** (1984), 595–627.
3. L. Block, J. Guckenheimer, M. Misiurewicz, and L.-S. Young, *Periodic points and topological entropy of one dimensional maps*, Lecture Notes in Math., vol. 819, Springer-Verlag, Berlin and New York, 1980, pp. 18–34.
4. W. A. Coppel, *Šarkovskii-minimal orbits*, Math. Proc. Cambridge Philos. Soc. **93** (1983), 397–408.
5. C.-W. Ho, *On the structure of the minimum orbits of periodic points for maps of the real line* (to appear).
6. M. Misiurewicz, *Horseshoes for mappings of the interval*, Bull. Acad. Polon. Sci. **27** (1979), 167–169.
7. Z. Nitecki, *Maps of the interval with closed periodic set*, Proc. Amer. Math. Soc. **85** (1982), 451–456.
8. ———, *Topological dynamics on the interval*, Progress in Math., vol. 21, Birkhäuser, Boston, Mass., 1982, pp. 1–73.
9. P. Stefan, *A theorem of Sharkovsky on the existence of periodic orbits of continuous endomorphisms of the real line*, Comm. Math. Phys. **54** (1977), 237–248.

INSTITUTE OF MATHEMATICS, ACADEMIA SINICA, TAIPEI, TAIWAN 115, REPUBLIC OF CHINA