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OPERATORS ON BANACH LATTICES AND THE RADON-NIKODÝM THEOREM

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ABSTRACT. For positive operators between Banach lattices a concept of absolute continuity is considered. On an AM-space with order unit norm, S is shown to be absolutely continuous with respect to T if and only if there is an approximation of S by finite sums of operators of the type $q \circ T \circ h$ where h and q are multiplication operators or orthomorphisms. Given T compact, compactness of S is characterized.

For positive operators S and T from a Dedekind complete Banach lattice E to a Banach lattice F, a concept of absolute continuity is considered. It is shown that for AM-spaces with order unit norm, S is "τ-absolutely continuous" with respect to T if and only if S can be approximated by finite sums of operators of the type $q \circ T \circ h$ where h and q are orthomorphisms on E and F respectively. The symbol "$\circ$" denotes composition so that for $f \in E$, $(q \circ T \circ h)(f) = q(T(h(f)))$. In particular E will be identified with functions on a topological space X and h will be identified with a multiplication by a function on X, again denoted by h, so that $(hf)(x) = h(x)f(x)$ for $x \in X$, and similarly for q. The operators τ-absolutely continuous with respect to T contain the operators less than or equal to a multiple of T. Theorem 3 relating compactness of S to that of T might be contrasted to a theorem of Dodds and Fremlin (see [**3** or **4**]) which states, in the presence of order continuous norms in E' and F, that the compactness of T implies the compactness of $S \leq T$.

Luxemburg (see [**7**, p. 185]) defined S to be *absolutely continuous with respect to* T if Sf is in the band generated by Tf for each $f \geq 0$ in E. This was studied in the setting of Dedekind complete Riesz spaces. We will consider the following analogue in the setting of Banach lattices.

We define S to be *topologically absolutely continuous with respect to* T (or τ-*absolutely continuous*) if for each positive element f in E the element of Sf is in the closure of the order ideal generated by Tf.

The condition S less than or equal to a multiple of T implies τ-absolutely continuous which in turn implies absolutely continuous (in the sense of Luxemburg).

Let $C(X)$ denote the Banach lattice of all continuous real-valued functions on a compact Hausdorff topological space X endowed with the supremum norm ($\| \|_\infty$). For linear functionals on $C(X)$, given X extremally disconnected (equivalently, $C(X)$ Dedekind complete), both concepts of absolute continuity coincide with the usual notion of absolute continuity for measures. The extremally disconnected assumption will be needed for the proof of the following proposition and in turn for

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subsequent theorems since τ-absolute continuity does not in general imply absolute continuity for measures. For example, consider a discrete measure and Lebesgue measure.

An operator is an *orthomorphism* on E if it is in the order ideal generated by the identity operator. Approximations by finite sums of operators of the type $q \circ T \circ h$ (the composition of the three operators) where h and q are orthomorphisms on E and F respectively have been used in [**5**, **6**, **9**]. Here we will compare absolute continuity to approximation by such operators.

We begin with the following characterization of τ-absolutely continuous.

PROPOSITION. *For two positive operators S and T from $C(X)$ to $C(Y)$, where X is extremally disconnected, the following statements are equivalent.*

(a) *S is τ-absolutely continuous with respect to T.*

(b) *For each positive f in E, if $Sf(y) > 0$ for a point y in Y, then $Tf(y) > 0$.*

(c) *If $Sf \geq g > 0$ and e is an order unit in $C(X)$, then for each $\varepsilon > 0$, there exists a $\lambda > 0$ such that $Tf \geq \lambda[g - \varepsilon e]$.*

(d) *For each positive linear functional ϕ on $C(Y)$ the measure $\phi \circ S$ is absolutely continuous with respect to $\phi \circ T$.*

PROOF. We first show that (a) implies (b). Let $Sf(y) > 0$. If $Tf(y) = 0$, then $Sf(y) = \lim[Sf \wedge nTf](y) = 0$, which is a contradiction. To show (b) implies (c), let $A = \{y\colon (g - \varepsilon e)(y) \geq 0\}$. Now A is a subset of $\{y\colon Tf(y) > 0\}$. Since A is compact, $Tf \geq \lambda(g - \varepsilon e)$ for some λ. To show (c) implies (a), note that (c) implies $Tf \geq \lambda(Sf - \varepsilon e)$. Thus $(Sf - \varepsilon e) \vee 0$ is in the order ideal generated by Tf, and hence Sf is in the closure of the ideal (in uniform norm). Clearly (d) implies (b). To show (b) implies (d), let (b) be satisfied and let $(\phi \circ S)(f) > 0$. Viewing ϕ as a measure, we have $\int S(f)\, d\phi > 0$ so that $\phi\{y\colon Sf(y) > 0\} > 0$. Hence $\phi\{y\colon Tf(y) > 0\} > 0$ and $(\phi \circ T)(f) > 0$.

We recall that orthomorphisms on $C(X)$ correspond to multiplication operators by elements of $C(X)$.

THEOREM 1. *Let S and T be positive linear maps from $C(X)$ into $C(Y)$, where X is extremally disconnected. S is τ-absolutely continuous with respect to T if and only if for each $f \geq 0$ in $C(X)$ and $\varepsilon > 0$, there exists positive orthomorphisms h_i on $C(X)$ and q_i on $C(Y)$ for a finite number of indices i, so that*

$$\left\| \left(S - \sum_i q_i \circ T \circ h_i \right)(f) \right\|_\infty < \varepsilon.$$

PROOF. Let S be τ-absolutely continuous with respect to T. For a fixed y in Y, the map assigning each f in $C(X)$ the value $T(f)(y)$ is a positive linear functional on $C(X)$. Thus, by the Riesz Representation Theorem, this functional corresponds to integration with respect to a measure μ_y (on X). Similarly, the functional defined by $S(f)(y)$ corresponds to a measure ν_y. Since the operator S is τ-absolutely continuous with respect to T, ν_y is absolutely continuous with respect to μ_y (see the Proposition). Hence, the Radon-Nikodým Theorem implies that

$$Sf(y) = \int f g_y \, d\mu_y$$

for a positive measurable function g_y on X. Since $S(1)(y)$ is finite, g_y is in L^1. Given $\varepsilon > 0$, there is a continuous function h_y in $C(X)$ satisfying $\|g_y - h_y\|_1 < \varepsilon$ (e.g., see [**1**, p. 203]). Now for f in $C(X)$,

$$|[T(h_y f) - S(f)]y| = \left|\int (h_y - g_y) f \, d\mu_y\right| < \varepsilon \|f\|_\infty.$$

Continuity implies that $|[T(h_y f) - S(f)]z| < \varepsilon\|f\|_\infty$ for all z in a neighborhood N_y of y. For each y in Y, let h_y and N_y be as above. Since Y is compact, there exists a finite partition of unity (e.g., see [**1**, p. 63]) subordinated to the cover of sets N_y (y in Y), i.e., there exist functions q_i $(1 \leq i \leq n)$ in $C(Y)$ such that $0 \leq q_i \leq 1$, q_i vanishes on the complement of some set N_y, and $\sum_{i=1}^n q_i = 1$. We set $h_i = h_y$ where q_i vanishes off of N_y. Let z be in Y and assume $q_i(z) \neq 0$. Now z is in an open set N_y corresponding to $q_i j$ so that $|S(f)z - T(fh_i)z| \leq \varepsilon\|f\|_\infty$. Thus

$$q_i(z)|S(f)z - T(fh_i)z| \leq \varepsilon\|f\|_\infty q_i(z),$$

and in turn

$$\left|S(f)z - \sum_{i=1}^n q_i(z)T(fh_i)z\right| \leq \varepsilon\|f\|_\infty.$$

Hence $\sum_{i=1}^n q_i \circ T \circ h_i$ is the desired approximation where h_i and q_i are now identified with the orthomorphisms of multiplication by these functions. Conversely, let $f \geq 0$ be in $C(X)$ and $(Tf)y = 0$. Then, assuming the approximation condition, for $\varepsilon > 0$ there exists an operator $S_\varepsilon = \sum q_i \circ T \circ h_i$ with the property that $|[(S - S_\varepsilon)(f)]y| < \varepsilon$. Since $S_\varepsilon(f)y = 0$, we conclude that $S(f)y = 0$.

We define S to be *order absolutely continuous* (or *o-absolutely continuous*) with respect to T if for each positive f in E, the element Sf is in the ideal generated by Tf.

Again for a functional on a Dedekind complete $C(X)$, this coincides with the usual concept of absolute continuity.

THEOREM 2. *Let S and T be positive operators from a Dedekind complete Banach lattice E having quasi-interior point e to a Banach lattice F. Let S be o-absolutely continuous with respect to T and denote by $\hat{F}$ the closure of the ideal generated by the range of T in F. For $\varepsilon > 0$ and $f \geq 0$ in the ideal generated by e, there exists a finite collection of orthomorphisms h_i on E and q_i on $\hat{F}$ so that $\|(S - \sum q_i \circ T \circ h_i)(f)\| < \varepsilon$ ($\|\cdot\|$ is the norm on F).*

PROOF. We identify E with a vector lattice of extended real-valued functions on an extremally disconnected compact space X (each function finite on a dense subset of X) where the ideal generated by e is identified with the lattice $C(X)$ and e is identified with the function 1 (e.g., see [**8**]). The Banach sublattice $\hat{F}$ of F contains a quasi-interior point $T(e)$. Thus we will identify $\hat{F}$ with a vector lattice of continuous extended real-valued functions on a compact space Y, where the order ideal generated by $T(e)$ is identified with $C(Y)$ and $T(e)$ is identified with the function 1. Now T and S restricted to $C(X)$ can be viewed as positive linear maps from $C(X)$ into $C(Y)$. Thus for f in $C(X)$ and $\varepsilon > 0$, Theorem 1 implies that there exists a finite sum $S_\varepsilon = \sum q_i \circ T \circ h_i$ so that $|(S_\varepsilon - S)(f)| < \varepsilon T(e)$ (for g in $\hat{F}$, $\|g\|_\infty < \varepsilon$ implies $|g| < \varepsilon T(e)$). It follows that $\|(S_\varepsilon - S)f\| < \varepsilon\|T(e)\|$, where

$\| \ \|$ is the norm in F. finally, each h_i and q_i can be viewed as an orthomorphism on all of E and $\hat{F}$ respectively (see [**10**]). For completeness, we include a verification of this fact. Let h be an element of $C(X)$ and $f \geq 0$ in E. The sequence $\{f_j\}$, where $f_j = f \wedge j1$, converges to f. Now h is bounded by a number M, so that

$$|h(f_j - f_i)| \leq M|f_j - f_i|.$$

This implies that $\|hf_j - hf_i\| \leq M\|f_j - f_i\|$ and thus the sequence $\{hf_j\}$ is Cauchy in the norm of E. The limit k of this sequence is greater than or equal to every hf_j. If k is not the least upper bound of the sequence $\{hf_j\}$, then there is an $r \geq 0$ in $C(Y)$ with $(k - hf_j) > r$ for all j. This implies $\|k - hf_j\| \geq \|r\|$, a contradiction. Given any f in E the function $f = f^+ - f^-$, so that h is an orthomorphism on E corresponding to multiplication by h.

THEOREM 3. *Let T be a compact positive operator between Banach lattices $C(X)$ and $C(Y)$, where X is extremally disconnected. A positive operator S, which is τ-absolutely continuous with respect to T, is compact if and only if S is the limit of operators of the type $\sum q_i \circ T \circ h_i$ in the operator norm, where q_i and h_i are orthomorphisms.*

PROOF. If S is the norm limit of operators of the type $\sum_{i=1}^{n} q_i \circ T \circ h_i$ (each of which is compact), then it is clear that S is compact. For the converse, assume that S is compact. Now for y in Y and $\varepsilon > 0$, let G denote the operator taking f to $(S(f) - T(h_y f))$ as described in the proof of Theorem 1. Now G is compact and $|G(f)(y)| < \varepsilon\|f\|_\infty$. We will show that there is a neighborhood N_y of y such that $G(f)(N_y)$ is contained in $(-3\varepsilon, 3\varepsilon)$ for all f in the unit ball of $C(X)$. Assume that this is not the case. Then there exists a net (y_α) convergent to y and functions f_α in the unit ball of $C(X)$ such that

(i) $G(f_\alpha)(y_\alpha) > 3\varepsilon$, and

(ii) $G(f_\alpha)(y) < \varepsilon$.

It follows from the compactness of the operator that a subnet of $G(f_\alpha)$ converges to some function h in $C(Y)$. Condition (ii) implies that h is less than $3\varepsilon/2$ on a neighborhood W of y while condition (i) implies that h exceeds 2ε on points in W, a contradiction. Thus there exists a neighborhood N_y of y so that $G(f)(N_y)$ is contained in $(-3\varepsilon, 3\varepsilon)$. Repeating the construction in the proof of Theorem 1, one obtains functions q_i and a partition of unity subordinated to the cover $\{N_y : y \in Y\}$. Then the operator norm of $(\sum q_i \circ T \circ h_i - S)$ is less than or equal to 3ε.

We cite several examples to indicate that the different types of absolute continuity discussed here are indeed distinct. S and T will denote operators on $C(\beta N)$ where βN denotes the Stone-Čech compactification of N.

First, let S denote the identity operator. Define the function g in $C(\beta N)$ by $g(n) = 1/n$ for n in N and extended to βN. We denote by T the operator of multiplication by g. S is absolutely continuous with respect to T in the sense of Luxemburg but S is not τ-absolutely continuous with respect to T.

Second, let T again denote the operator of multiplication by the function g as above and let S denote the operator of multiplication by h, where h is defined at each n by $h(n) = 1/\sqrt{n}$ and extended to βN. S is τ-absolutely continuous with respect to T but not o-absolutely continuous with respect to T.

Third, let S denote the identity operator and T the operator defined by $T(f) = \sum_{n\in N}(f(n)/n^2)\cdot\hat{1}$, where $\hat{1}$ is the function with constant value one. Here, S is o-absolutely continuous with respect to T but S is not less than or equal to a multiple of T.

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