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RANDOM REALS AND SOUSLIN TREES

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ABSTRACT. It is consistent that there are no Souslin trees in any random extension of V; thus, the continuum can be singular of cofinality ω_1 with Souslin's hypothesis holding.

In [**4**], Roitman proved that adding a Cohen real causes $\mathrm{MA}_{\aleph_1}$ to fail: If g is a Cohen real over V, then in $V[g]$ there is a ccc partial ordering P such that $P \times P$ does not have the ccc (see also Galvin [**3**], where the existence of such a P is derived from CH). Shelah [**5**] improved Roitman's result to: If g is a Cohen real over V, then in $V[g]$ there is a Souslin tree. Kunen (see [**4**]) noted that Roitman's theorem also holds for random extensions: If r is random over V, then a ccc P, with $P \times P$ not ccc, exists in $V[r]$. Left open was the random real analogue of Shelah's theorem, that is, whether it is a theorem of ZFC that adding a random real adds a Souslin tree.

The question whether adding a Sacks real adds a Souslin tree is independent of ZFC+SH. Namely, suppose s is a Sacks real over V. Carlson showed (unpublished) that if V satisfies a fragment of axiom $\mathrm{A} + 2^{\aleph_0} > \aleph_1$ (which is consistent relative to the consistency of ZFC), then $V[s]$ satisfies $\mathrm{MA}_{\aleph_1}$, and the author (unpublished) showed that if V satisfies CH, then $V[s]$ satisfies $\diamondsuit_{\omega_1}$.

The result of this paper is that if $\mathrm{MA}_{\aleph_1}$ holds, then the adjunction of any number of random reals does not add a Souslin tree. Recall that if MA_κ holds, then any tree T of size $\leq \kappa$ with no paths of length ω_1 is special, that is, T is the union of countably many antichains (Baumgartner, Malitz, and Reinhardt [**1**]).

THEOREM. *If* MA_κ *holds and* $\mathcal{B}$ *is a measure algebra, then in* $V^{\mathcal{B}}$ *every tree of size* $\leq \kappa$ *with no paths of length* ω_1 *is special.*

This theorem fills in the last part to the result that the continuum can be arbitrary with Souslin's hypothesis holding. That Souslin's hypothesis can hold with the continuum being an arbitrary regular cardinal, or an arbitrary singular cardinal of cofinality greater than ω_1, is the work of Solovay and Tennenbaum [**7**] and Jensen (see [**2**]). To get that $2^{\aleph_0}$ can be any singular cardinal of cofinality ω_1 with the Souslin hypothesis holding, begin with a model of $\mathrm{MA}_{\aleph_1}$ and add λ random reals (λ any cardinal satisfying $\operatorname{cf} \lambda = \omega_1$ and $\lambda^{\aleph_0} = \lambda$).

Also, by starting with a model of $\mathrm{MA}_{\aleph_1}$ and adding at least measurably many random reals, the consistency that there is a real valued measurable cardinal $\leq 2^{\aleph_0}$ (see Solovay [**6**]) and Souslin's hypothesis holds, is obtained, relative to the consistency of a measurable cardinal.

PROOF OF THE THEOREM. Assume MA_κ. Suppose $\mathcal{B}$ is an atomless measure algebra via a countably additive measure $\mu\colon \mathcal{B} \to [0,1]$. Suppose T is a term in

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the forcing language of $\mathcal{B}$ for a tree of the form $(\kappa, <_T)$, and suppose T is forced to have no paths of length ω_1. A $\mathcal{B}$-term F will be constructed such that it is forced that F is a partition of T into $\aleph_0$-many antichains. The first two lemmas show that the poset, used to produce F by an application of Martin's axiom, has the ccc. Fix a uniform ultrafilter $\mathcal{U}$ on ω_1.

LEMMA 1. *Suppose* $0 < \gamma < \varepsilon$, $X \in \mathcal{U}$, $b_\beta \in \mathcal{B}$ $(\beta \in X)$, *and each* $\mu(b_\beta) \geq \varepsilon$. *Then for some* $c \in \mathcal{B}$, $\mu(c) > 0$, *and for all* $c' \leq c$, $\{\beta\colon \mu(c' \wedge b_\beta) \geq \gamma \cdot \mu(c')\} \in \mathcal{U}$.

PROOF. If for every c there is a counterexample $c' \leq c$, take a partition of $\mathbf{1}$ into such counterexamples. Let C be a finite subset of the partition such that $\mu(\bigvee C) \geq 1 - (\varepsilon - \gamma)$. For each $c' \in C$, $X_{c'} = \{\beta\colon \mu(c' \wedge b_\beta) < \gamma \cdot \mu(c')\} \in \mathcal{U}$. Pick $\beta \in \bigcap_{c' \in C} X_{c'}$. Then $\mu(b_\beta) < \gamma \cdot \mu(\bigvee C) + (\varepsilon - \gamma) \leq \varepsilon$, a contradiction.

LEMMA 2. *Suppose* $\varepsilon > 0$, $X \in [\omega_1]^{\aleph_1}$, $X_\alpha \in \mathcal{U}$ $(\alpha \in X)$, *and* $\mu(b_{\alpha\beta}) \geq \varepsilon$ $(\alpha \in X, \beta \in X_\alpha)$. *Define, in* $V[G_{\mathcal{B}}]$, $R(\alpha, \bar{\alpha}) \Leftrightarrow$ *for some* β, $b_{\alpha\beta} \wedge b_{\bar{\alpha}\beta} \in G_{\mathcal{B}}$. *Then for some* $c > 0$ *in* $\mathcal{B}$,

$$c \Vdash \exists Y \in [X]^{\aleph_1} \forall \alpha, \bar{\alpha} \in Y \ R(\alpha, \bar{\alpha}).$$

PROOF. Pick $\gamma < \varepsilon$. For each $\alpha \in X$ apply Lemma 1 to the sequence $b_{\alpha\beta}$ $(\beta \in X_\alpha)$, getting a $c_\alpha \in \mathcal{B}$. Then by the ccc-ness of $\mathcal{B}$ there is a $c \in \mathcal{B}$ forcing that $W = \{\alpha\colon c_\alpha \in G_{\mathcal{B}}\}$ has size $\aleph_1$.

We claim that c works. Namely pick N so that $N \cdot \gamma > 1$. Using $\aleph_1 \to (\aleph_1, N)^2$ in $V[G_{\mathcal{B}}]$, either there is a Y as in the lemma or there is an $e \in \mathcal{B}, e \leq c$, and $\alpha_0 < \alpha_1 < \cdots < \alpha_{N-1}$ such that $e \Vdash$ each $\alpha_i \in W$ and $\neg R(\alpha_i, \alpha_j)$ $(i < j \leq N-1)$. Suppose such an e exists. Then $e \leq c_{\alpha_0} \wedge \cdots \wedge c_{\alpha_{N-1}}$, so by the choice of the c_{α_i}'s there are $X_{\alpha_i} \in \mathcal{U}$ such that for $\beta \in X_{\alpha_i}$, $\mu(b_{\alpha_i\beta} \wedge e) \geq \gamma \cdot e$. Pick $\beta \in \bigcap_{i<N} X_{\alpha_i}$. Since $N \cdot \gamma > 1$, $e \wedge b_{\alpha_i\beta} \wedge b_{\alpha_j\beta} > 0$ for some $i < j < N$. But $e \wedge b_{\alpha_i\beta} \wedge b_{\alpha_j\beta}$ forces $R(\alpha_i, \alpha_j)$, a contradiction.

A poset $\mathcal{P}$ for an application of MA_κ will be constructed. For $\alpha < \kappa$, let A_α be the countable set $\{\delta\colon [|\delta <_T \alpha|] > 0\}$.

Define complete subalgebras $\mathcal{B}_\alpha$ of $\mathcal{B}$ by induction on $\alpha < \kappa$. $\mathcal{B}_0$ is an arbitrary countably generated atomless subalgebra of $\mathcal{B}$. For $0 < \alpha < \kappa$, let $\mathcal{B}_\alpha$ be the complete subalgebra of $\mathcal{B}$ generated by

$$\bigcup_{\delta \in A_\alpha} (\mathcal{B}_\delta \cup \{[|\delta <_T \alpha|]\}).$$

Then each $\mathcal{B}_\alpha$ is isomorphic to the countably generated atomless measure algebra. Fix a countable dense $\mathcal{C}_\alpha \subseteq \mathcal{B}_\alpha - \{0\}$.

A condition in $\mathcal{P}$ is a function f with domain of the form $E \times W$, where $E \in [\kappa]^{<\aleph_0}$ and $W \in [\omega]^{<\aleph_0}$, such that

(1) $f(\alpha, n) \in \mathcal{B}_\alpha$, and either $f(\alpha, n) = 0$ or $\mu(f(\alpha, n)) > \frac{1}{2}$.

(2) $f(\alpha, n) \wedge f(\beta, n) \Vdash \alpha$ and β are T-incomparable. (If $f(\alpha, n) \wedge f(\beta, n) = 0$, we say that this forcing relation holds vacuously.)

For $f, g \in \mathcal{P}$, g extends f if $E_f \subseteq E_g$ and $W_f \subseteq W_g$, and for each $(\alpha, n) \in \operatorname{dom} f, f(\alpha, n) \geq g(\alpha, n)$ and if $f(\alpha, n) > 0$, then $g(\alpha, n) > 0$. Thus $\mathcal{P}$ adds a sequence $d_{\alpha,n}$ $(\alpha < \kappa, n < \omega)$ where $d_{\alpha,n} \in \mathcal{B}$ is either 0 or is of measure $\frac{1}{2}$ and obtained by amoeba forcing on $\mathcal{B}_\alpha$.

Let $f \sim g$ denote that f and g are compatible. If $n \in W_f$ let $f \upharpoonright n = f \upharpoonright E_f \times \{n\}$. Evidently, if $W_f = W_g$ and $f \upharpoonright n \sim g \upharpoonright n$, for each $n \in W_f$, then $f \sim g$.

LEMMA 3. *$\mathcal{P}$ has the ccc.*

PROOF. Suppose $f_\gamma \in \mathcal{P}$ $(\gamma < \omega_1)$ and $f_\gamma \not\sim f_{\gamma'}$ $(\gamma \neq \gamma')$. We may assume that all the W_γ's are the same set W, and by the Δ-system lemma we may assume that all the E_γ's have size n, and that for some E and all $\gamma \neq \gamma', E_\gamma \cap E_{\gamma'} = E$. We may further assume that for some $\varepsilon > 0$ and all γ, if $\mu(f_\gamma(\langle\alpha, n\rangle)) > \frac{1}{2}$, then $\mu(f_\gamma(\langle\alpha, n\rangle)) \geq \frac{1}{2} + \varepsilon$. For $\alpha \in E$, approximate each $f_\gamma(\langle\alpha, n\rangle)$ by finite Boolean combinations of members of $\mathcal{C}_\alpha$ to within accuracy $\varepsilon/3$. We may assume that these approximations are independent of γ, and thus that $f_\gamma \restriction E \times W \sim F_{\gamma'} \restriction E \times W$. So we may assume $E = \emptyset$.

Since $\mathrm{Card}\{\beta\colon [|\beta <_T \alpha|] > 0\} \leq \aleph_0$ for each $\alpha < \kappa$, we may assume that if $\gamma < \gamma', \beta \in E_{\gamma'}$, and $\alpha \in E_\gamma$, then $[|\beta <_T \alpha|] = 0$. If $f_\gamma \not\sim f_{\gamma'}$, then for some k, $f_\gamma \restriction E_\gamma \times \{k\} \not\sim f_{\gamma'} \restriction E_{\gamma'} \times \{k\}$. Pick a k such that

$$\{\gamma\colon \{\gamma'\colon f_\gamma \restriction E_\gamma \times \{k\} \not\sim f_{\gamma'} \restriction E_\gamma \times \{k\}\} \in \mathcal{U}\} \in \mathcal{U}.$$

Let $\bar{f}_\gamma = f_\gamma \restriction E_\gamma \times \{k\}$, and write $E_\gamma = \{\sigma_{\gamma,0}, \sigma_{\gamma,1}, \ldots, \sigma_{\gamma,n-1}\}$.

Given $\gamma < \gamma'$ with $\bar{f}_\gamma \not\sim \bar{f}_{\gamma'}$, there must be $i, j < n$ with $\mu[|\sigma_{\gamma,i} <_T \sigma_{\gamma',j}|] \geq \varepsilon/n+1$. Namely, suppose for all i, j that $\mu[|\sigma_{\gamma,i} <_T \sigma_{\gamma',j}|] < \varepsilon/n+1$. Then $\bar{f}_\gamma$ and $\bar{f}_{\gamma'}$ would be extended by $\bar{f}$, where $\mathrm{dom}\, f = (E_\gamma \cup E_{\gamma'}) \times \{k\}$, $\bar{f} \restriction E_\gamma \times \{k\} = \bar{f}_\gamma$, and for $j < n$,

$$\bar{f}(\langle\sigma_{\gamma',j}, k\rangle) = \bar{f}_{\gamma'}(\langle\sigma_{\gamma',j}, k\rangle) - \bigvee_{i<n} [|\dot{\sigma}_{\gamma,i} <_T \sigma_{\gamma',j}|]$$

(this is an element of $\mathcal{B}_{\sigma_{\gamma',j}}$).

Thus there are $i, j < n$ such that $\{\gamma\colon \{\gamma'\colon \mu(b_{\gamma\gamma'}) \geq \varepsilon/n + 1\} \in \mathcal{U}\} \in \mathcal{U}$, where $b_{\gamma\gamma'} = [|\sigma_{\gamma,i} <_T \sigma_{\gamma',j}|]$. Apply Lemma 2 to the $b_{\gamma\gamma'}$'s, getting a generic extension $V[G_\mathcal{B}]$ in which for some $X \in [\omega_1]^{\aleph_1}$, for every $\gamma, \bar{\gamma}$ in X, there is a γ' with $b_{\gamma\gamma'} \wedge b_{\bar{\gamma}\gamma'} \in G_\mathcal{B}$.

In this extension, if $\gamma, \gamma' \in X$ and $\gamma < \gamma'$, then $\sigma_{\gamma,i} <_T \sigma_{\gamma',j}$ since T is a tree. This contradiction completes the proof of Lemma 3.

Recall that $\mathcal{C}_\alpha$ is a countable dense subset of $\mathcal{B}_\alpha - \{0\}$. For $\alpha < \kappa$, $c \in \mathcal{C}_\alpha$, let $\mathcal{D}_{\alpha,c} = \{f \in \mathcal{P}\colon \exists n \langle\alpha, n\rangle \in \mathrm{dom}\, f,\ \mu(f(\langle\alpha, n\rangle)) > \frac{1}{2},\ \mu(f(\langle\alpha, n\rangle) - c) < \frac{1}{2} - \frac{1}{2}\mu(c)\}$. Since $\mathcal{B}_\alpha$ is atomless, each $\mathcal{D}_{\alpha,c}$ is cofinal in $\mathcal{P}$. Let $\mathcal{G}$ be $\mathcal{P}$-generic over the $\mathcal{D}_{\alpha,c}$'s.

Let $d_{\alpha,n} = \bigwedge\{f(\langle\alpha, n\rangle\colon \langle\alpha, n\rangle \in \mathrm{dom}\, f,\ f \in \mathcal{G}\}$. In $V[G_\mathcal{B}]$ let $A_n = \{\alpha\colon d_{\alpha n} \in G_\mathcal{B}\}$. We claim that it is forced in $\mathcal{B}$ that the A_n's are T-antichains and $\bigcup_{n<\omega} A_n = \kappa$. They are antichains by the definition of $\mathcal{P}$. Now suppose $b \Vdash \alpha \notin \bigcup_{n<\omega} A_n$. Since each $d_{\alpha n} \in \mathcal{B}_\alpha$, we may as well assume $b \in \mathcal{B}_\alpha$. Pick $c \in \mathcal{C}_\alpha$ with $c - b < \frac{1}{4}\mu(c)$. Pick $f \in \mathcal{G}$ and $n < \omega$ with $\mu(f(\langle\alpha, n\rangle)) > \frac{1}{2}$ and $\mu(f(\langle\alpha, n\rangle) - c) < \frac{1}{2}$. Then $b \wedge d_{\alpha n} > 0$ and forces that $\alpha \in A_n$. This completes the claim and the theorem.

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