ABSTRACT. Let $M$ be an $(8k + 2)$-dimensional closed spin manifold and $N$ an orientable hypersurface of $M$ with the induced spin structure. If $M$ admits a metric with positive scalar curvature and $N$ represents a nonzero homology class of $H_{8k+1}(M; \mathbb{Z})$, then the KO-characteristic number $a(N)$ vanishes. This result relates to the conjecture by Gromov and Lawson on the vanishing of higher $\hat{A}$-genera.

1. Introduction and statement of the Theorem. Scalar curvature is one of the simplest invariants of Riemannian manifolds; however, we know that there need be some topological conditions for a manifold to admit a Riemannian metric with positive scalar curvature. For instance, Hitchin [3] proved that the exotic sphere $\Sigma$ with nonvanishing KO-characteristic number $a(\Sigma)$ does not admit such a metric although the standard sphere definitely admits one.

Moreover there is a topological obstruction, the so-called higher $\hat{A}$-genus, to admitting a metric with positive scalar curvature (see [2]). The definition is as follows.

Let $M$ be a closed spin manifold and let $u$ be a rational cohomology class of $K(\pi, 1)$. Given a homomorphism from $\pi_1(M)$ to $\pi$, we obtain the corresponding map

$$f : M \to K(\pi, 1).$$

Consider the number

$$\hat{A}(u)(M) = \langle \hat{A}(M)f^* (u), [M] \rangle.$$

We call this number a higher $\hat{A}$-genus of $M$ associated with $u$. Gromov and Lawson [2] proved the following.

THEOREM [2]. Let $M$ be a closed spin manifold of even dimension. If $M$ admits a metric with positive scalar curvature, then the higher $\hat{A}$-genus $\hat{A}(u)(M)$ of $M$ vanishes for $\pi = \mathbb{Z}^k$ and for each $u \in H^*(T^k; \mathbb{Q})$.

We note that the Theorem also holds for odd-dimensional manifolds. The proof is obtained by considering $M \times S^1$.

In particular, for a generator $u$ of $H^k(T^k; \mathbb{Q})$, we obtain

$$\hat{A}(u)(M) = \langle \hat{A}(M)f^*(u), [M] \rangle = \langle \hat{A}(M), f^*(u) \cap [M] \rangle$$

$$= \langle \hat{A}(N), [N] \rangle = \hat{A}(N),$$

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where $N$ is a submanifold of $M$ whose Poincaré dual is $f^*(u)$. Thus, in this case the higher $\hat{A}$-genus $\hat{A}(u)(M)$ can be regarded as a $\hat{A}$-genus of $N$ and hence the Theorem says that the $\hat{A}$-genus $\hat{A}(N)$ must vanish.

In this paper we prove a similar result for the $KO$-characteristic number $\alpha$.

**Theorem.** Let $M$ be an $(8k + 2)$-dimensional closed spin manifold and $N$ an orientable hypersurface of $M$ which represents a nonzero homology class of $H_{8k+1}(M; \mathbb{Z})$. We fix the spin structure on $M$ and equip $N$ with the induced spin structure. Then, if $M$ admits a Riemannian metric with positive scalar curvature, the $KO$-characteristic number $\alpha(N)$ vanishes.

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2. Gysin homomorphisms and $KO$-characteristic numbers. We begin with the definition of $KO$-characteristic numbers. To do this we shall review Gysin homomorphisms for $KO^*$-theory.

Let $X$ be an $(n + i)$-dimensional closed manifold and let $M$ be an $n$-dimensional submanifold with an embedding $f : M \subset X$. We assume that the normal bundle $\nu$ of $M$ is a spin vector bundle. Then, $\nu \times \mathbb{R}^{8k-i}$ is also a spin vector bundle of rank $8k$ and hence $KO^*(M)$ is isomorphic to $KO^{*+8k}(\nu \times \mathbb{R}^{8k-i})$ by the Thom isomorphism. Furthermore a natural inclusion $\nu \times \mathbb{R}^{8k-i} \subset X \times \mathbb{R}^{8k-i}$ induces a homomorphism from $KO^*(\nu \times \mathbb{R}^{8k-i})$ to $KO^*(X \times \mathbb{R}^{8k-i})$ and $KO^*(X \times \mathbb{R}^{8k-i})$ is isomorphic to $KO^*_{-8k+i}(X)$ by the suspension isomorphism. Composing these homomorphisms, we obtain

$$KO^*(M) \to KO^{*+8k}(\nu \times \mathbb{R}^{8k-i}) \to KO^{*+8k}(X \times \mathbb{R}^{8k-i}) \to KO^{*+i}(X),$$

which is called a Gysin homomorphism induced by $f$ and denoted by $f_!$.

Here, we set the Euclidean space $\mathbb{R}^{8m}$ as $X$ and consider an embedding $M \subset \mathbb{R}^{8m}$. Then a spin structure on $M$ makes $\nu$ into a spin vector bundle and under the identification of $KO^*(\mathbb{R}^{8m})$ with $KO^*_{-8m}(\text{point})$, a $KO$-characteristic number $\alpha(M)$ is defined by

$$\alpha(M) = f_!(1) \in KO^{-n}(\text{point}),$$

where $1 \in KO^0(M)$. We note that the definition depends only on the spin structure on $M$ and is independent of the choice of embeddings by virtue of the Bott periodicity theorem.

Since $KO^{-8m-1}(\text{point})$ and $KO^{-8m-2}(\text{point})$ are isomorphic to $\mathbb{Z}_2$, $KO^{-8m}(\text{point})$ and $KO^{-8m-4}(\text{point})$ are isomorphic to $\mathbb{Z}$, and the others are zero, we sometimes consider that $\alpha(M)$ takes a value in $\mathbb{Z}_2$ or $\mathbb{Z}$. In particular $\alpha(M)$ is equal to $\hat{A}(M)$ for $n = 8m$.

3. Vanishing theorem and index theorem. In this section we quote two theorems we need for the proof.

**Vanishing Theorem** [3]. Let $M$ be a closed spin manifold and $E$ a real vector bundle over $M$ of rank $m$. We denote by $D^E$ the Dirac operator with coefficient bundle $E$, namely,

$$D^E : \Gamma(S \otimes E) \to \Gamma(S \otimes E),$$
where $S$ denotes the spinor bundle over $M$. Then, if $E$ admits a flat $O(m)$-connection and $M$ admits a Riemannian metric with positive scalar curvature, the operator $D^E$ has trivial kernel.

**Atiyah-Singer Index Theorem** [1]. Let $M$ be an $n$-dimensional closed spin manifold with $n = 8k + 2$ or $8k + 1$ and let $E$ be a real vector bundle over $M$. Then the following statements hold.

1. The kernel of $D^E$ has the structure of a finite-dimensional complex vector space for $n = 8k + 2$ and has the structure of a finite-dimensional real vector space for $n = 8k + 1$.
2. The image of $E$ by the Gysin homomorphism $f_!$ in §2 is given by

$$f_!(E) = \dim_{\mathbb{C}} \ker D^E \mod 2 \quad \text{for } n = 8k + 2,$$

$$f_!(E) = \dim_{\mathbb{R}} \ker D^E \mod 2 \quad \text{for } n = 8k + 1,$$

where $f_!(E)$ is considered as taking a value in $\mathbb{Z}_2$.

We note that Hitchin's result [3] quoted earlier is obtained by combining these theorems.

**4. Proof of the Theorem.** Let $M$ and $N$ be such manifolds as described in the statement of the Theorem. Then, we may assume that there is a smooth map $\varphi: M \to S^1$ such that $\varphi$ is transverse with $S^1$ at some point $x_0 \in S^1$ and the inverse image $\varphi^{-1}(x_0)$ is $N$. Denote by $\eta$ the canonical generator of $KO^{-1}(\text{point})$. Then the following diagram commutes by virtue of naturality of the Thom isomorphism:

$$\begin{array}{ccc}
KO^*(x_0) & \xrightarrow{(\varphi|_M)^*} & KO^*(N) \\
i_! & \downarrow & \downarrow j_! \\
KO^{*+1}(S^1) & \xrightarrow{\varphi^*} & KO^{*+1}(M) \\
\eta_! & \downarrow & \eta_! \\
KO^*(S^1) & \xrightarrow{\varphi^*} & KO^*(M)
\end{array}$$

where $i: x_0 \subset S^1$ and $j: N \subset M$. In particular we obtain

$$\eta \cdot (j_!(1)) = (\varphi^* \eta \cdot i_!(1)).$$

Now let $u$ be the generator of $H^1(S^1: \mathbb{Z}_2)$. Since the isomorphism classes of real line bundles are classified by the 1st Stiefel-Whitney class $w_1$, there is a real line bundle $L$ over $S^1$ such that $w_1(L) = u$. Then it follows that

$$\eta \cdot i_!(1) = [L] - 1 \quad \text{in } KO^0(S^1).$$

Thus we obtain

$$\eta \cdot \alpha(N) = \eta \cdot (f \circ j_!(1)) = \eta \cdot f_!(j_!(1)) = f_!(\eta \cdot j_!(1)) = f_!(\varphi^*(\eta \cdot i_!(1))) = f_!(\varphi^*([L] - 1)) = f_!(\varphi^*[L]) - f_!(1),$$

where $f$ is an embedding of $M$ into $\mathbb{R}^{8m}$ as in §2. Note that $\varphi^*(L)$ admits a flat $O(1)$-connection. Thus, if $M$ admits a Riemannian metric with positive scalar curvature.
curvature, it follows that $\ker D\varphi^*(L) = 0$ and $\ker D = 0$ from the vanishing theorem. Hence, by the Atiyah-Singer index theorem we obtain that $f_1(\varphi^*[L]) = 0$ and $f_1(1) = 0$. It is known that the multiplication by $\eta$ from $KO^{-n+1}(\text{point})$ to $KO^{-n}(\text{point})$ is an isomorphism for $n = 8k + 2$. Thus the Theorem follows from the fact that $\eta \cdot \alpha(N) = 0$.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, UNIVERSITY OF TOKYO, HONGO, TOKYO 113, JAPAN

Current address: Department of Mathematics, Brown University, Providence, Rhode Island 02912