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POSITIVE SCALAR CURVATURE AND KO-CHARACTERISTIC NUMBERS

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ABSTRACT. Let M be an $(8k+2)$-dimensional closed spin manifold and N an orientable hypersurface of M with the induced spin structure. If M admits a metric with positive scalar curvature and N represents a nonzero homology class of $H_{8k+1}(M;\mathbf{Z})$, then the KO-characteristic number $\alpha(N)$ vanishes. This result relates to the conjecture by Gromov and Lawson on the vanishing of higher $\hat{A}$-genera.

1. Introduction and statement of the Theorem. Scalar curvature is one of the simplest invariants of Riemannian manifolds; however, we know that there need be some topological conditions for a manifold to admit a Riemannian metric with positive scalar curvature. For instance, Hitchin [**3**] proved that the exotic sphere Σ with nonvanishing KO-characteristic number $\alpha(\Sigma)$ does not admit such a metric although the standard sphere definitely admits one.

Moreover there is a topological obstruction, the so-called higher $\hat{A}$-genus, to admitting a metric with positive scalar curvature (see [**2**]). The definition is as follows.

Let M be a closed spin manifold and let u be a rational cohomology class of $K(\pi,1)$. Given a homomorphism from $\pi_1(M)$ to π, we obtain the corresponding map

$$f\colon M \to K(\pi,1).$$

Consider the number

$$\hat{A}(u)(M) = \langle \hat{\mathfrak{A}}(M) f^*(u), [M]\rangle.$$

We call this number a higher $\hat{A}$-genus of M associated with u. Gromov and Lawson [**2**] proved the following.

THEOREM [**2**]. *Let M be a closed spin manifold of even dimension. If M admits a metric with positive scalar curvature, then the higher $\hat{A}$-genus $\hat{A}(u)(M)$ of M vanishes for $\pi = \mathbf{Z}^k$ and for each $u \in H^*(T^k;\mathbf{Q})$.*

We note that the Theorem also holds for odd-dimensional manifolds. The proof is obtained by considering $M \times S^1$.

In particular, for a generator u of $H^k(T^k;\mathbf{Q})$, we obtain

$$\begin{aligned}\hat{A}(u)(M) &= \langle \hat{\mathfrak{A}}(M) f^*(u), [M]\rangle = \langle \hat{\mathfrak{A}}(M), f^*(u)\cap [M]\rangle \\ &= \langle \hat{\mathfrak{A}}(N), [N]\rangle = \hat{A}(N),\end{aligned}$$

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where N is a submanifold of M whose Poincaré dual is $f^*(u)$. Thus, in this case the higher $\hat{A}$-genus $\hat{A}(u)(M)$ can be regarded as a $\hat{A}$-genus of N and hence the Theorem says that the $\hat{A}$-genus $\hat{A}(N)$ must vanish.

In this paper we prove a similar result for the KO-characteristic number α.

THEOREM. *Let M be an $(8k+2)$-dimensional closed spin manifold and N an orientable hypersurface of M which represents a nonzero homology class of $H_{8k+1}(M;\mathbf{Z})$. We fix the spin structure on M and equip N with the induced spin structure. Then, if M admits a Riemannian metric with positive scalar curvature, the KO-characteristic number $\alpha(N)$ vanishes.*

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2. Gysin homomorphisms and KO-characteristic numbers. We begin with the definition of KO-characteristic numbers. To do this we shall review Gysin homomorphisms for KO^*-theory.

Let X be an $(n+i)$-dimensional closed manifold and let M be an n-dimensional submanifold with an embedding $f\colon M \subset X$. We assume that the normal bundle ν of M is a spin vector bundle. Then, $\nu \times \mathbf{R}^{8k-i}$ is also a spin vector bundle of rank $8k$ and hence $KO^*(M)$ is isomorphic to $KO^{*+8k}(\nu \times \mathbf{R}^{8k-i})$ by the Thom isomorphism. Furthermore a natural inclusion $\nu \times \mathbf{R}^{8k-i} \subset X \times \mathbf{R}^{8k-i}$ induces a homomorphism from $KO^*(\nu \times \mathbf{R}^{8k-i})$ to $KO^*(X \times \mathbf{R}^{8k-i})$ and $KO^*(X \times \mathbf{R}^{8k-i})$ is isomorphic to $KO^{*-8k+i}(X)$ by the suspension isomorphism. Composing these homomorphisms, we obtain

$$KO^*(M) \to KO^{*+8k}(\nu \times \mathbf{R}^{8k-i}) \to KO^{*+8k}(X \times \mathbf{R}^{8k-i}) \to KO^{*+i}(X),$$

which is called a Gysin homomorphism induced by f and denoted by $f_!$.

Here, we set the Euclidean space $\mathbf{R}^{8m}$ as X and consider an embedding $M \subset \mathbf{R}^{8m}$. Then a spin structure on M makes ν into a spin vector bundle and under the identification of $KO^*(\mathbf{R}^{8m})$ with $KO^{*-8m}(\text{point})$, a KO-characteristic number $\alpha(M)$ is defined by

$$\alpha(M) = f_!(1) \in KO^{-n}(\text{point}),$$

where $1 \in KO^0(M)$. We note that the definition depends only on the spin structure on M and is independent of the choice of embeddings by virtue of the Bott periodicity theorem.

Since $KO^{-8m-1}(\text{point})$ and $KO^{-8m-2}(\text{point})$ are isomorphic to $\mathbf{Z}_2$, $KO^{-8m}(\text{point})$ and $KO^{-8m-4}(\text{point})$ are isomorphic to $\mathbf{Z}$, and the others are zero, we sometimes consider that $\alpha(M)$ takes a value in $\mathbf{Z}_2$ or $\mathbf{Z}$. In particular $\alpha(M)$ is equal to $\hat{A}(M)$ for $n = 8m$.

3. Vanishing theorem and index theorem. In this section we quote two theorems we need for the proof.

VANISHING THEOREM [3]. *Let M be a closed spin manifold and E a real vector bundle over M of rank m. We denote by D^E the Dirac operator with coefficient bundle E, namely,*

$$D^E\colon \Gamma(S \otimes E) \to \Gamma(S \otimes E),$$

where S denotes the spinor bundle over M. Then, if E admits a flat $O(m)$-connection and M admits a Riemannian metric with positive scalar curvature, the operator D^E has trivial kernel.

ATIYAH-SINGER INDEX THEOREM [**1**]. *Let M be an n-dimensional closed spin manifold with $n = 8k+2$ or $8k+1$ and let E be a real vector bundle over M. Then the following statements hold.*

(1) *The kernel of D^E has the structure of a finite-dimensional complex vector space for $n = 8k+2$ and has the structure of a finite-dimensional real vector space for $n = 8k+1$.*

(2) *The image of E by the Gysin homomorphism $f_!$ in §2 is given by*

$$f_!(E) = \dim_{\mathbf{C}} \ker D^E \mod 2 \quad \textit{for } n = 8k+2,$$
$$f_!(E) = \dim_{\mathbf{R}} \ker D^E \mod 2 \quad \textit{for } n = 8k+1,$$

where $f_!(E)$ is considered as taking a value in $\mathbf{Z}_2$.

We note that Hitchin's result [**3**] quoted earlier is obtained by combining these theorems.

4. Proof of the Theorem. Let M and N be such manifolds as described in the statement of the Theorem. Then, we may assume that there is a smooth map $\varphi\colon M \to S^1$ such that φ is transverse with S^1 at some point $x_0 \in S^1$ and the inverse image $\varphi^{-1}(x_0)$ is N. Denote by η the canonical generator of $KO^{-1}(\text{point})$. Then the following diagram commutes by virtue of naturality of the Thom isomorphism:

$$\begin{array}{ccc}
KO^*(x_0) & \xrightarrow{(\varphi|_N)^*} & KO^*(N) \\
{\scriptstyle i_!}\downarrow & & {\scriptstyle j_!}\downarrow \\
KO^{*+1}(S^1) & \xrightarrow{\varphi^*} & KO^{*+1}(M) \\
{\scriptstyle \eta\cdot}\downarrow & & {\scriptstyle \eta\cdot}\downarrow \\
KO^*(S^1) & \xrightarrow{\varphi^*} & KO^*(M)
\end{array}$$

where $i\colon x_0 \subset S^1$ and $j\colon N \subset M$. In particular we obtain

$$\eta \cdot (j_!(1)) = \varphi^*(\eta \cdot i_!(1)).$$

Now let u be the generator of $H^1(S^1\colon \mathbf{Z}_2)$. Since the isomorphism classes of real line bundles are classified by the 1st Stiefel-Whitney class w_1, there is a real line bundle L over S^1 such that $w_1(L) = u$. Then it follows that

$$\eta \cdot i_!(1) = [L] - 1 \quad \text{in } KO^0(S^1).$$

Thus we obtain

$$\begin{aligned}
\eta \cdot \alpha(N) &= \eta \cdot (f \circ j)_!(1) = \eta \cdot f_!(j_!(1)) \\
&= f_!(\eta \cdot j_!(1)) = f_!(\varphi^*(\eta \cdot i_!(1))) \\
&= f_!(\varphi^*([L] - 1)) = f_!(\varphi^*[L]) - f_!(1),
\end{aligned}$$

where f is an embedding of M into $\mathbf{R}^{8m}$ as in §2. Note that $\varphi^*(L)$ admits a flat $O(1)$-connection. Thus, if M admits a Riemannian metric with positive scalar

curvature, it follows that $\ker D^{\varphi^*(L)} = 0$ and $\ker D = 0$ from the vanishing theorem. Hence, by the Atiyah-Singer index theorem we obtain that $f_!(\varphi^*[L]) = 0$ and $f_!(1) = 0$. It is known that the multiplication by η from $KO^{-n+1}(\text{point})$ to $KO^{-n}(\text{point})$ is an isomorphism for $n = 8k + 2$. Thus the Theorem follows from the fact that $\eta \cdot \alpha(N) = 0$.

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