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THE NONCOMPACT HYPERBOLIC 3-MANIFOLD OF MINIMAL VOLUME

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ABSTRACT. We utilize maximal cusp volumes in order to prove that the Gieseking manifold is the unique complete noncompact hyperbolic 3-manifold of minimal hyperbolic volume.

1. Introduction. By a hyperbolic 3-manifold, we mean a complete Riemannian 3-manifold of constant sectional curvature -1. Each such manifold M can be realized as H^3/Σ where Σ is a discrete torison-free subgroup of the isometries of H^3. We are allowing orientation-reversing isometries and hence nonorientable 3-manifolds. All of the hyperbolic 3-manifolds that are considered here have finite volume.

As much of the work that has been done on hyperbolic 3-manifolds has been focused on the orientable case, we will make some remarks about how the basic theorems of hyperbolic 3-manifold theory extend to nonorientable manifolds. As a consequence of Mostow's Rigidity Theorem, we know that hyperbolic volume is a topological invariant for finite volume hyperbolic 3-manifolds. Jorgensen and Thurston [**10**] proved that the set of volumes of orientable hyperbolic 3-manifolds is well ordered and of order type ω^ω. Since any nonorientable hyperbolic 3-manifold is double-covered by an orientable hyperbolic 3-manifold, we know that the set of volumes of all hyperbolic 3-manifolds is also well ordered. In particular, there is a hyperbolic 3-manifold of minimum volume among all hyperbolic 3-manifolds and a noncompact hyperbolic 3-manifold of minimum volume among all noncompact hyperbolic 3-manifolds.

In Chapter 5 of [**10**], Thurston discusses the fact that a finite volume noncompact orientable hyperbolic 3-manifold can be cut open along two-sided tori, leaving a compact component and a finite set of cusps, each homeomorphic to $T^2 \times [0,1)$. Again, because a nonorientable hyperbolic 3-manifold is double-covered by an orientable hyperbolic 3-manifold, a finite volume noncompact nonorientable hyperbolic 3-manifold can be cut open along two-sided tori and two-sided Klein bottles, leaving a compact component and a finite set of cusps, each homeomorphic to either $T^2 \times [0,1)$ or $K^2 \times [0,1)$.

There are many Dehn surgeries that can be performed on an orientable cusp $T^2 \times [0,1)$, corresponding to various choices for surgery coefficients, and Thurston proves in Chapter 6 of [**10**] that all of the resultant manifolds which are hyperbolic will have smaller volume than the original manifold. There is, however, only one

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way to glue a solid Klein bottle to a nonorientable cusp $K^2 \times [0,1)$ and the resulting manifold need not be hyperbolic. Hence we do not know that for a given n-cusp nonorientable hyperbolic 3-manifold, there exists an $(n-1)$-cusp nonorientable hyperbolic 3-manifold of smaller volume as is true in the orientable case.

Utilizing an examination of maximal cusp volumes in a noncompact hyperbolic 3-manifold in conjunction with the work of R. Meyerhoff [**6**], we prove that the Gieseking manifold, a nonorientable 3-manifold which will be defined in the next section, is the unique noncompact hyperbolic 3-manifold of minimal volume and consequently that the minimum volume of a noncompact hyperbolic 3-manifold is exactly the volume of an ideal regular tetrahedron in H^3, that is $v = 1.01494\ldots$.

We note that although R. Meyerhoff [**7**] has already discovered the minimal volume noncompact hyperbolic 3-orbifold, questions that remain to be solved include determining the minimum volume for the set of all hyperbolic 3-manifolds, the set of all hyperbolic 3-orbifolds and the set of noncompact orientable hyperbolic 3-manifolds (see [**6**] for the current status on these questions). Minimum volumes for n-cusp hyperbolic 3-manifolds for given n are investigated in [**1**].

In the following, we will always utilize the upper half-space for hyperbolic 3-space.

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2. The Gieseking manifold. This 3-manifold was first described by Gieseking [**5**] in 1912. Let T be an ideal regular tetrahedron in H^3, that is, a tetrahedron will all four vertices on the sphere at infinity and all dihedral angles $\pi/3$. Then, as in Figure 1, identify faces A to A' and B to B' so that the orientations on the edges match up correctly. Such identifications can be performed by hyperbolic isometries, each orientation-reversing. These isometries generate a discrete torsion-free subgroup of the group of all isometries of H^3. After the identifications, all six edges will be identified and the sum of the angles about this one edge will now add up to 2π.

The resulting manifold is a noncompact hyperbolic 3-manifold, with volume $v = 1.01494\ldots$ (see [**9**] for the calculation of the volume of an ideal regular tetrahedron). We can draw the link diagram (defined in Chapter 4 of [**10**]), two possibilities for which appear in Figure 2. Note that the Gieseking manifold is double covered by the figure-eight knot complement. Thurston conjectured in [**10**] that the figure-eight knot complement is one of two noncompact orientable hyperbolic 3-manifolds of minimal volume. This conjecture, the truth of which would imply that the Gieseking manifold is a noncompact hyperbolic 3-manifold of minimal volume, motivated our interest in this question.

3. Maximal cusp volumes. Let M be a noncompact finite volume hyperbolic 3-manifold which has been decomposed into a compact component and a finite set of cusps, each homeomorphic to either $T^2 \times [0,1)$ or $K^2 \times [0,1)$. If we lift any such cusp to H^3, we obtain an infinite set of disjoint horoballs.

Examining first the situation when M has exactly one cusp, lift that cusp to the corresponding set of disjoint horoballs, each of which is the image of any other by some group element. Expand the horoballs equivariantly until two first become tangent. We call the projection of thses expanded horoballs back to M the maximal

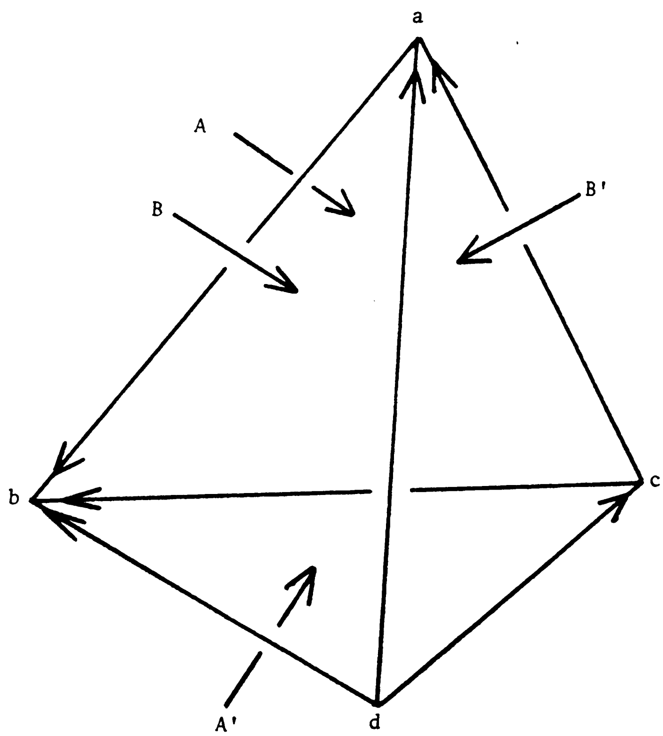

FIGURE 1

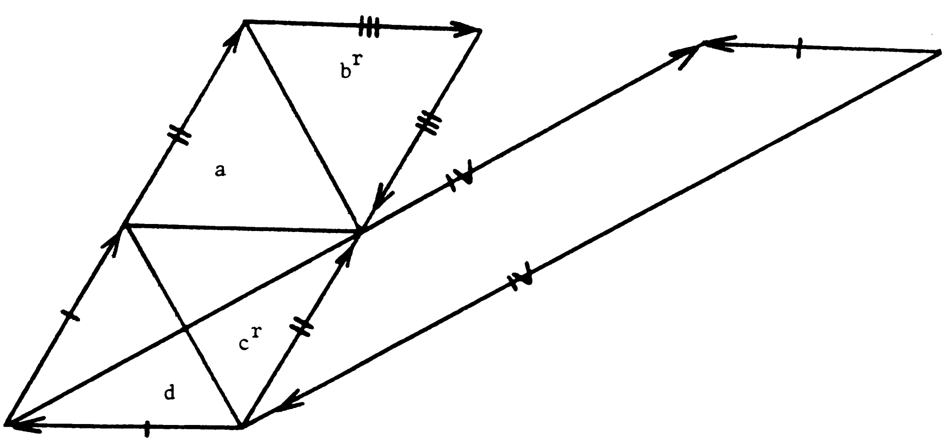

FIGURE 2

cusp of M. We can assume one such horoball H is centered about $\{\infty\}$. Let h be the Euclidean height of the boundary horosphere of H above the x-y plane. Let P be the subgroup of hyperbolic isometries in $\pi_1(M)$ which fix $\{\infty\}$. Then the volume of this maximal cusp, denoted $v_C(M)$, is simply the volume of a fundamental region in H for the action on H by P. This is given by the Euclidean area of the link diagram divided by $2h^2$.

In the case that M has more than one cusp, we define a maximal volume for each cusp C exactly as above. Note that the maximal cusps in a manifold can intersect. Further details on computing maximal cusp volumes will appear in [**2**].

We first prove a theorem that has been known for some time and appears explicitly in the orientable case in [**8**].

THEOREM 1. *$v_C(M) \geq \sqrt{3}/4$ for any single cusp C in a finite volume hyperbolic 3-manifold M.*

PROOF. Let C be a maximal cusp in M which lifts to an infinite set of horoballs with disjoint interiors in H^3. We may assume one such horoball H_1 is centered about $\{\infty\}$ and a second such horoball H_2 is tangent to H_1, centered about $\{0\}$, and has Euclidean diameter h.

Let P again be the subgroup of $\pi_1(M)$ that fixes $\{\infty\}$. Since P is either the fundamental group of a torus or a Klein bottle, there exists a fundamental domain D for the action of P on the x-y plane which is a parallelogram such that one of its vertices occurs at $\{0\}$ and such that all four of its vertices are identified by P. Each of the vertices of D is then the center of a horoball which covers C and has Euclidean diameter h. Since these horoballs must have disjoint interiors, each vertex must be a distance h from each of the other three vertices. Hence we can center a disk of radius $h/2$ about each vertex of D such that the interiors of any two of the disks are disjoint. Then, since the images of D under the action of P tile the plane, the images of these disks under the action of P will form a disk-packing in the plane. As the densest disk-packing in the plane is the hexagonal packing, the ratio of the area of a disk to the area of D is at most $\pi/2\sqrt{3}$. Hence the area of D is at least $\sqrt{3}h^2/2$ and $v_C(M) \geq \sqrt{3}/4$. □

This extimate can be improved by a factor of two.

THEOREM 2. *$v_C(M) \geq \sqrt{3}/2$ for any single cusp C in a finite volume hyperbolic 3-manifold M.*

PROOF. Let C, H_1, H_2, P and D be as described in the previous proof. Then because H_1 and H_2 both cover C, there must exist a group element g sending H_2 to H_1 and hence sending 0 to ∞.

Suppose $g(\infty) = p(0)$ for some p in P. Then $p^{-1}g(\infty) = 0$ and $p^{-1}g(0) = \infty$. Hence $p^{-1}g$ sends the geodesic running from 0 to ∞ back to itself but reverses its orientation. Thus $p^{-1}g$ fixes a point in H^3, contradicting the fact it is a nontrivial covering translation.

Hence $g(\infty)$ must be some point y not contained in $P(0)$. Since H_1 is tangent to H_2, $g(H_1)$ must be tangent to $g(H_2) = H_1$. Thus $g(H_1)$ is a horoball of the same Euclidean radius as H_2 but not contained in the set of horoballs $P(H_2)$. Therefore D must contain a point, for convenience call it y, such that every pair of points in $P(0) \cup P(y)$ are at least a distance h apart.

Centering disks of radius $h/2$ about each of the vertices of D and about y and then letting P act on these will again result in a disk-packing of the plane. However, now D must contain the equivalent of two disks. Hence, as in the previous proof, the area of D is at least $\sqrt{3}h^2$ and $v_C(M) \geq \sqrt{3}/2$. □

COROLLARY 3. *The Gieseking manifold has the minimum volume among all noncompact hyperbolic 3-manifolds.*

PROOF. Let M be any finite volume noncompact hyperbolic 3-manifold, with any positive number of cusps. Meyerhoff [6] utilized results on horoball-packings by K. Boroczky [4] in order to prove that for a 1-cusp hyperbolic 3-manifold M, the total hyperbolic volume of M, denote vol(M), satisfies the inequality $\mathrm{vol}(M) \geq v_C(M)/(\sqrt{3}/2v)$. This result holds true if C is any single cusp in M. Theorem 2 then implies that M has volume greater than or equal to $v = 1.01494\ldots$ and the Gieseking manifold realizes this lower bound. Note that if $\mathrm{vol}(M) = v$ then $v_C(M) = \sqrt{3}/2$ for every cusp in M. □

4. Uniqueness.

THEOREM 4. *If M is a noncompact hyperbolic 3-manifold of minimal volume, then M is the Gieseking manifold.*

PROOF. M must have volume $1.01494\ldots$ and each cusp of M must have maximal volume $\sqrt{3}/2$. Let C be a maximal cusp in M and let H be a horoball centered about $\{\infty\}$ covering C. Let P, D, and y be as in the previous proof. In order that the maximal disk-packing density in the plane is attained, it must be the case that the disks of radius $h/2$ centered about $P(0) \cup P(y)$ are in a hexagonal packing. Hence, we have a tiling of the plane by equilateral triangles of edge length h such that the vertices of the triangles are exactly $P(0) \cup P(y)$.

Let $\{t_1, \ldots, t_n\}$ be the finite subset of these triangles such that $\mathring{t}_i \cap D \neq \emptyset$ for $i = 1, \ldots, n$. Let $D' = \bigcup_{i=1}^{n} t_i$. Then the images of D' under P cover the entire x-y plane. For each i, let T_i be the ideal regular tetrahedron which has for its vertices the three vertices of t_i and $\{\infty\}$.

Let $R = \bigcup_{i=1}^{n} T_i$. We claim that the images of R under $\pi_1(M)$ will cover all of H^3. It is clear that the images of R under $\pi_1(M)$ will cover all of the maximal horoballs corresponding to C since the images of R clearly cover H and all of the maximal horoballs for C are identified by $\pi_1(M)$. It suffices to show that the bottom face f_i on each of the tetrahedra T_i is identified to some other face on one of the tetrahedra by some element of $\pi_1(M)$, as we then have a tiling of H^3 by ideal regular tetrahedra. However, since the three vertices corresponding to f_i are centers for three pairwise tangential horoballs all corresponding to C and there exists a group element μ sending a particular one of these three vertices to $\{\infty\}$, μ must send the remaining two vertices to the centers of a pair of tangential horoballs of Euclidean diameter h corresponding to C. Hence μ sends f_i to a face of $p(T_j)$ for some p in P and some j. Thus $p^{-1}\mu$ sends f_i to a face of T_j, as desired.

It is also true that no interior point of any one tetrahedron T_i is identified to any other point of T_i by an element of $\pi_1(M)$ since any such map must permute the maximal tangential horoballs corresponding to C and hence be a permutation on the vertices of T_i. But such an isometry would fix the central point of T_i, contradicting the fact it must be a covering translation.

Since $\mathrm{vol}(M) = \mathrm{vol}(T_1)$ and $\pi_1(M)$ must leave the set of centers of horoballs invariant, the other $n-1$ tetrahedra must all be identified to T_1, implying T_1 is a fundamental region for $\pi_1(M)$. Hence our fundamental region is a single ideal regular tetrahedron with pairs of faces identified. It is simple to check that the only such identifications which yield a 3-manifold produce the Gieseking manifold. □

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