PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 100, Number 4, August 1987

LIOUVILLIAN SOLUTIONS OF THE DIFFERENTIAL EQUATION $y'' + S(x)y = 0$ WITH $S(x)$ BINOMIAL

MINORU SETOYANAGI

ABSTRACT. If a differential equation $y'' + (ax^p + bx^q)y = 0$ with $p > q$ has a liouvillian solution, then p is an even number $2m$ and the number $s = (m+1)/(p-q)$ is an integer. The case $s = 2$ occurs only if $m = 1$.

0. Introduction. Let K be an algebraically closed field of characteristic 0, and let $K(x)$ be the field of rational functions of x over K. We define $x' = 1$ and $c' = 0$ for $c \in K$, and $K(x)$ becomes a differential field. Consider a differential equation

$$y'' + (ax^p + bx^q)y = 0, \qquad ab \neq 0,\ p > q,$$

where a, b are constants and p, q are nonnegative integers.

THEOREM. *If our equation has a liouvillian solution, then p is an even number $2m$ and the number $s = (m+1)/(p-q)$ is an integer. The case $s = 2$ occurs only if $m = 1$.*

This improves the main results of R. R. Hailperin (formerly R. M. Roberts) [**1**], which is unpublished. The case $s = 1$ will be discussed in §3.

Our theorem will be proved in §2 by the following criterion due to H. P. Rehm [**5**]:

A differential equation $y'' + S(x)y = 0$ with S a polynomial is reducible over $K(x)$ if it has a liouvillian solution. In this case $-S = P^2 + P' + R$ with P, R polynomials of x and $\deg R \leq \deg P - 1$. The equation $y'' - (P^2 + P' + R)y = 0$ is reducible over $K(x)$ if and only if $z'' + 2Pz' - Rz = 0$ has a polynomial solution.

Rehm applied it to the case where $S(x)$ is a quadratic polynomial and obtained a necessary and sufficient condition for it to have a liouvillian solution.

For liouvillian solutions of $y'' + S(x)y = 0$ with $S(x)$ a rational function we have several results: for Bessel's equation due to Liouville, for Gauss' hypergeometric equation due to F. Klein, M. Hukuhara, S. Ōhasi, and T. Kimura. For the bibliography confer with M. Matsuda's book [**3**].

The author would like to express his sincere gratitude to Professor M. Matsuda for his kind guidance.

ACKNOWLEDGMENT. Our contributions to the Theorem are first, to simplify Hailperin's proof by Rehm's criterion, and second, to remove her assumption that our equation is reducible. Her results will be given in §3.

1. Rehm's criterion. Our statement is not given directly by Rehm. The first part will be proved as follows (cf. M. Matsuda [**4**]). We shall show that if our equation has a liouvillian solution, then it is reducible over $K(x)$. To the contrary,

Received by the editors February 3, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 12H05, 13N05, 34A05.

suppose that our equation, having a liouvillian solution, is irreducible over $K(x)$. There is no algebraic solution. We have a rational function $a(x)$ over K which satisfies

$$a'' = 3aa' + 2S' - 4aS - a^3 \tag{1}$$

(cf. Kaplansky [**2**, §25]). It takes the form

$$a(x) = \sum \frac{e}{x-c} \neq 0, \qquad c, e \in K,\ e \neq 0$$

(cf. M. Matsuda [**3**, p. 97]). Comparing the coefficients of $(x-c)^{-3}$ in (1) we have $2e = -3e^2 - e^3$, and e is either -1 or -2. We indicate $\deg S$ and its leading coefficient by m and A respectively. Let us multiply both sides of (1) by x^{1-m} and set $x = \infty$. Then we have

$$0 = A\left(2m - 4\sum e\right),$$

but this is impossible.

A proof of the second part will be given as follows. Suppose that our equation is reducible over $K(x)$. Then there is a rational function $v(x)$, which is the logarithmic derivative of a solution y, satisfying $v' + v^2 = -S$. It takes the form $v = P + Q'/Q$ with P, Q polynomials. Hence,

$$P^2 + P' + Q''/Q + 2PQ'/Q = -S,$$

and we have

$$(Q'' + 2PQ')/Q = R \tag{2}$$

with R a polynomial. Thus, $S = -P^2 - P' - R$ and $z = Q$ is a solution of $z'' + 2Pz' = Rz$. Conversely, if S takes this form with (2), then the solution y of $y'/y = P + Q'/Q$ satisfies our equation $y'' + S(x)y = 0$, and it is reducible over $K(x)$.

2. Proof of the Theorem. Since $\deg R \leq \deg P - 1$, we have

$$\deg S = \deg P^2 = 2\deg P, \qquad S = ax^p + bx^q.$$

Hence, p is an even number. Let us set

$$P = ax^m + a'x^{m'} + \cdots + a^{(u)}x^{m^{(u)}} + \cdots,$$

where $m > m' > \cdots > m^{(u)} > \cdots \geq 0$, $aa' \cdots \neq 0$, and

$$R = cx^{m-1} + c_1x^{m-2} + \cdots, \qquad c \neq 0.$$

Let a polynomial

$$Q = t_0x^n + t_1x^{n-1} + \cdots, \qquad t_0 \neq 0,$$

be a solution of $z'' + 2Pz' - Rz = 0$. Comparing the coefficients of x^{n+m-1}, we have

$$c = 2na. \tag{3}$$

First, suppose that $P(x)$ is a monomial. Then

$$P^2 + P' + R = a^2x^{2m} + (m+2n)ax^{m-1} + c_1x^{m-2} + \cdots$$

by (3). Since $(m+2n)a \neq 0$, we have

$$P^2 + P' + R = a^2x^{2m} + (m+2n)ax^{m-1}.$$

Hence, $p = 2m$, $q = m-1$, and $s = 1$.

Second, suppose that $P(x)$ is not a monomial. Then we have

$$2m > m + m' > m^{(u)} + m^{(v)}, \qquad u + v > 1.$$

Hence, $p = 2m$, $q = m + m'$, and

$$s = (m+1)/(p-q) = (m+1)/(m-m'). \tag{4}$$

It is greater than one. Since $S(x)$ is a binomial, it takes the form

$$-S = a^2x^{2m} + 2aa'x^{m+m'}.$$

Let us define k_u by

$$m - m^{(u)} = k_u(m - m'), \qquad u > 0.$$

By induction in u we shall show that k_u is an integer. By definition, $k_1 = 1$. Since $m + m^{(u+1)} > m - 1$, we have

$$m + m^{(u+1)} = m^{(v)} + m^{(w)}, \qquad 0 < v \leq w < u + 1.$$

Therefore,

$$m - m^{(u+1)} = m - m^{(v)} + m - m^{(w)} = (k_v + k_w)(m - m')$$

and $k_{u+1} = k_v + k_w$. Hence, k_u is an integer. There exist indices u and v for which $m - 1 = m^{(u)} + m^{(v)}$. If $m-1$ were not equal to $m^{(u)} + m^{(v)}$ for any pair of u and v, then we would have $(m+2n)a = 0$ by (3), which is impossible. We have

$$m + 1 = m - m^{(u)} + m - m^{(v)} = (k_u + k_v)(m - m').$$

Thus s is an integer by (4). The first part of our theorem has been proved.

Assume that $s = 2$. Then

$$P = ax^m + bx^r, \qquad ab \neq 0,\ r = (m-1)/2.$$

If $P(x)$ were not a binomial, then $2m' = m + m''$ and $m - 1 = m + m''$, which is impossible. Since $P^2 + P' + R$ is a binomial, we have

$$R = cx^{m-1} + dx^{r-1}, \qquad c = -ma - b^2,\ d = -rb.$$

By (3) we have

$$a = -b^2/(2n+m). \tag{5}$$

Let us set $j = (m+1)/2$. Comparing the coefficients of $x^{n-i+m-1}$ in $Q'' + 2PQ' - RQ = 0$, we have

$$(n-i+m+1)(n-i+m)t_{i-2j} + [2b(n-i+j) - d]t_{i-j} + [2a(n-i) - c]t_i = 0$$

for $0 \leq i \leq n+m+1$. Here we assume that $t_i = 0$ if either $i < 0$ or $n < i$. Let us examine it separately:

(i) For $1 \leq i \leq j-1$, we have $[2(n-i)a - c]t_i = -2iat_i = 0$ and $t_1 = t_2 = \cdots = t_{j-1} = 0$.

(ii) For $j \le i \le n$, t_i is determined inductively by

$$2iat_i = [2(n-i)+m+j]bt_{i-j} + (n-i+m+1)(n-i+m)t_{i-2j}. \tag{6}$$

Hence we obtain $t_i = 0$ if $i \not\equiv 0 \pmod{j}$.

(iii) For $n+1 \le i \le n+j$, we have

$$0 = [2(n-i)+m+j]bt_{i-j} + (n-i+m+1)(n-i+m)t_{i-2j}. \tag{7}$$

(iv) For $n+j < i \le n+m-1$, we have

$$(n-i+m+1)(n-i+m)t_{i-2j} = 0$$

and

$$t_{n-j+1} = \cdots = t_{n-2} = 0. \tag{8}$$

For $i = hj$ $(0 \le h \le n/j)$, t_{hj} takes the form $(-1/b)^h A_h t_0$ with A_h a rational number. Here,

$$A_0 = 1, \qquad A_1 = (2n+m)(2n+r)/(m+1).$$

By (5) and (6), we have

(9)

$$\begin{aligned} A_h &= [(2n+m)/2hj] \\ &\quad \times [\{2n-(2h-3)j-1\}A_{h-1} - \{n-(h-2)j\}\{n-(h-2)j-1\}A_{h-2}] \end{aligned}$$

for $1 < h \le n/j + 2$. Here we assume that $A_h = 0$ if $n/j < h$. We shall show that

$$A_h > [n-(h-1)j]A_{h-1}, \qquad h \le n/j, \tag{10}$$

by induction in h. For $h = 1$, we have

$$[n-(1-1)j]A_0 = n < A_1.$$

By the assumption of induction we obtain

$$\begin{aligned} A_h &= [(2n+m)/2hj] \\ &\quad \times [\{2n-(2h-3)j-1\}A_{h-1} - \{n-(h-2)j\}\{n-(h-2)j-1\}A_{h-2}] \\ &> [2n-(2h-3)j-1]A_{h-1} - [n-(h-2)j-1]A_{h-1} \\ &= [n-(h-1)j]A_{h-1}. \end{aligned}$$

Thus inequality (10) has been proved. We have $A_h > 0$, $0 \le h \le n/j$, because

$$n-(h-1)j = n-hj+j > 0, \qquad h \le n/j,$$

in (10). Hence, we have $t_{hj} \ne 0$ $(0 \le h \le n/j)$ and $n \equiv 0, 1 \pmod{j}$ by (8). First suppose that $n = kj$. By (10) we have

$$A_k > jA_{k-1}. \tag{11}$$

On the other hand, we have

$$0 = (j-1)A_k - j(j-1)A_{k-1}, \qquad h = k+1,$$

by (9). Assume that $m \ne 1$. Then $j-1 = (m-1)/2 \ne 0$ and we have $A_k = jA_{k-1}$ which contradicts (11). Secondly, suppose that $n-1 = kj$. By (10) we have

$$A_k > [kj+1-(k-1)j]A_{k-1} = (j+1)A_{k-1}. \tag{12}$$

On the other hand, by (9) for $h = k + 1$ we have

$$0 = (j+1)A_k - (j+1)jA_{k-1}.$$

Since $j + 1 \neq 0$, we have $A_k = jA_{k-1}$ which contradicts (12).

For the case $m = 1$, see H. P. Rehm [**5**].

REMARK. If the square of a polynomial $P(x)$ of degree m satisfies the condition that

$$P^2 = a^2x^{2m} + 2abx^{2m-j} + A(x), \qquad \deg A < m,$$

then it takes the form

$$P = ax^m \sum \binom{1/2}{h} (2bx^{-j}/a^2)^h, \qquad 0 \leq h \leq m/j.$$

This remark was given by M. Matsuda.

3. Hailperin's results. The first part of our theorem was obtained by R. R. Hailperin. She proved the reducibility directly by the result in Kaplansky's book [**2**] quoted in §1. For the second part of our theorem she obtained a partial result that in case $s = 2$ we have $m = 1$ if our equation is reducible. Her paper contains the following proofs.

1. Hailperin proves the integrality of the number $s = (m+1)/(p-q)$ as follows. If our equation has a liouvillian solution then there exists a polynomial $Q(x)$ whose roots are simple such that

$$(P + Q'/Q)^2 + (P + Q'/Q)' = -(\alpha x^{2m} + \beta x^{2m-j}). \tag{13}$$

First we shall show that $j \leq m + 1$. If $P(x)$ is a monomial, $P = ax^m$, then $p = 2m$, $q = m - 1$, and $j = p - q = m + 1$. If $P(x)$ is not a monomial, $P = ax^m + a'x^{m'} + \cdots$, then

$$-(\alpha x^{2m} + \beta x^{2m-j}) = a^2x^{2m} + 2aa'x^{m+m'}$$

and

$$2m - j = m + m'.$$

Hence, $j = m - m' < m + 1$. Next we shall show that j divides $m + 1$. Let a polynomial $P = ax^m + a_1x^{m-1} + \cdots + a_m$ be a solution of (13). Comparing the coefficients of x^k $(m \leq k \leq 2m)$ we have

$$a_0^2 = -\alpha, \qquad \sum_{p+q=i} a_pa_q = 0, \quad i \neq j,\ 1 \leq i \leq m, \qquad \sum_{p+q=j} a_pa_q = -\beta,$$

and $a_i = 0$ if $i \not\equiv 0 \pmod{j}$ and $1 \leq i \leq m$. Comparing the coefficients of x^{m-1} we obtain

$$\sum_{p+q=m+1} a_pa_q + (2n+m)a_0 = 0.$$

If j does not divide $m + 1$, then the first term $\sum a_pa_q$ vanishes and it contradicts our assumption that $a_0 \neq 0$.

2. Hailperin proves that $j = 1$ as follows. Let us define c_{hg} $(1 \leq h \leq k+2, 1 \leq g \leq k+1)$ by

$$\begin{aligned} c_{h,h-1} &= (2n+m)(n+m+1-i)(n+m-i), \\ c_{h,h} &= -(2n+m)[2(n-i)+3j-1], \\ c_{h,h+1} &= 2i, \end{aligned}$$

and $c_{hg} = 0$ for the other case, where $i = hj$ and k is the greatest integer such that $kj \leq n$. Then

$$\begin{aligned}
&c_{11}A_0 + c_{12}A_1 = 0,\\
&c_{h,h-1}A_{h-2} + c_{h,h}A_{h-1} + c_{h,h+1}A_h = 0, \qquad 2 \leq h \leq k,\\
&c_{k+1,k}A_{k-1} + c_{k+1,k+1}A_k = 0,\\
&c_{k+2,k+1}A_k = 0.
\end{aligned}$$

Since $A_0 \neq 0$, we have $\prod c_{h,h-1} = 0$ $(1 < h \leq k+2)$, and $n \equiv 0, 1 \pmod{j}$. It follows that $c_{k+2,k+1} = 0$. Hence

$$\det(c_{hg}) = 0, \qquad 1 < h, g \leq k+1.$$

Here we have $c_{h,h-1} \equiv c_{h,h+1} \equiv 0 \pmod{j}$, $c_{h,h} \equiv -1 \pmod{j}$, and j should be one.

3. Hailperin discusses the case $s = 1$ as follows. By (4), $P(x)$ is a monomial, $P = ax^m$. We have $R = cx^{m-1}$. Comparing the coefficients of $x^{m-i+m-1}$ in $Q'' + 2PQ' + RQ = 0$, we have

$$(n-i+m+1)(n-i+m)t_{i-m-1} + [2a(n-i) - c]t_i = 0$$

for $0 \leq i \leq m+n+1$, where $t_i = 0$ if either $i < 0$ or $n < i$. Let us examine separate cases:

(i) For $0 < i \leq m$, we have $2ait_i = 0$ from (3) and $t_1 = t_2 = \cdots = t_m = 0$.

(ii) For $m < i \leq n$, t_i is determined inductively by

$$2ait_i = (n-i+m+1)(n-i+m)t_{i-m-1}$$

from (3). We have $t_i = 0$ if and only if $i \equiv 0 \pmod{m+1}$.

(iii) For $n < i \leq n+m-1$, we have

$$(n-i+m+1)(n-i+m)t_{i-m-1} = 0$$

and

$$t_{n-m} = t_{n-m+1} = \cdots = t_{n-2} = 0.$$

Hence, $n \equiv 0, 1 \pmod{m+1}$. Therefore, we obtain the following result: The equation $y'' + (\alpha x^{2m} + \beta x^{m-1})y = 0$ with α, β constants has a liouvillian solution if and only if there exists an integer n such that

$$(m+2n)^2\alpha + \beta^2 = 0, \qquad n \equiv 0, 1 \pmod{m+1}.$$

BIBLIOGRAPHY

1. R. R. Hailperin (formerly R. M. Roberts), *On the solvability of a second order linear homogeneous differential equation*, Doctoral Dissertation, Univ. of Pennsylvania, 1960.
2. I. Kaplansky, *An introduction to differential algebra*, Hermann, Paris, 1957.
3. M. Matsuda, *Lectures on algebraic solutions of hypergeometric differential equations*, Lectures in Math., Dept. of Math., Kyoto Univ., No 15, Kinokuniya, Tokyo, 1985.
4. ——, *Liouvillian solutions of second order differential equation without Fuchsian singularities*, Nagoya Math. J. **103** (1986), 145–148.
5. H. P. Rehm, *Galois groups and elementary solutions of some linear differential equations*, J. Reine Angew. Math. **307** (1979), 1–7.

DEPARTMENT OF MATHEMATICS, MAIZURU TECHNICAL COLLEGE, 234 SHIRATA, MAIZURU, KYOTO-FU, JAPAN