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$*$-PURE SUBGROUPS OF COMPLETELY DECOMPOSABLE ABELIAN GROUPS

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ABSTRACT. In this note we prove that

(i) homogeneous $*$-pure subgroups of completely decomposable groups are completely decomposable,

(ii) $*$-pure subgroups of finite rank completely decomposable groups are almost completely decomposable.

We also characterize those reduced completely decomposable groups G, with $T(G)$ satisfying the maximum condition, any $*$-pure subgroup of which is also completely decomposable.

In [**2**], a subgroup H of a torsion-free group G is said to be *$*$-balanced* if H is balanced in G and

$$\text{(i)} \qquad \langle H^*(\tau)\rangle_* = H \cap \langle G^*(\tau)\rangle_*$$

for every type τ. We shall say that H is *$*$-pure* in G if H is pure in G and equation (i) above is satisfied for every type τ.

Throughout this note, unless otherwise specified, G will denote a completely decomposable abelian group and its extractable typeset will be denoted by $\mathcal{E}(G)$. We let $G = \bigoplus_{\tau\in\mathcal{E}(G)} G_\tau$ be a homogeneous decomposition of G and, for every $\tau \in \mathcal{E}(G)$, $\pi_\tau\colon G \to G_\tau$ is the projection such that $\ker \pi_\tau = \bigoplus_{\tau'\neq\tau} G_{\tau'}$.

In [**4**] we proved that homogeneous pure subgroups of G are also completely decomposable provided that $\mathcal{E}(G)$ is countable. The following theorem implies that the countability condition of $\mathcal{E}(G)$ can be relaxed if we consider homogeneous $*$-pure subgroups of G.

THEOREM 1. *Homogeneous $*$-pure subgroups of completely decomposable groups are completely decomposable.*

PROOF. Let H be a homogeneous $*$-pure subgroup of a completely decomposable group G. Let $0 \neq h \in H$ and suppose that $\operatorname{type}_G(h) = \tau_0$. Then $h = \sum_{\tau\geq\tau_0} \pi_\tau(h)$. If $\tau_0 \notin \mathcal{E}(G)$, then

$$h = \sum_{\tau>\tau_0} \pi_\tau(h) \in H \cap G^*(\tau_0) = \langle H^*(\tau_0)\rangle_* = \{0\},$$

a contradiction. Thus $\tau_0 \in \mathcal{E}(G)$ and $\pi_{\tau_0}(h) \neq 0$. Hence $H \cong \pi_{\tau_0}(H) \subseteq G_{\tau_0}$. Since $\pi_{\tau_0}(H)$ is homogeneous of type τ_0 and is a subgroup of a homogeneous completely decomposable group G_{τ_0}, also of type τ_0, it is completely decomposable by Lemma 86.6 in [**3**].

The following example illustrates that the homogeneity condition on H in the previous theorem is necessary.

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EXAMPLE. Let $G = G_1 \oplus G_2 \oplus G_3$ be a torsion-free group, where G_i is of rank one and type τ_i and $G_i \ncong Q$, the group of rationals, $1 \leq i \leq 3$. Assume that τ_1 and τ_2 are incomparable and $\tau_3 \geq \sup\{\tau_1, \tau_2\} > \tau_i$, $i = 1, 2$. By Lemma 1 in [**1**], G contains an indecomposable pure subgroup H with typeset $T(H) = \{\tau_0, \tau_1, \tau_2\}$, where $\tau_0 = \inf\{\tau_1, \tau_2\}$. Since τ_i is maximal in $T(H)$,

$$\langle H^*(\tau_i)\rangle_* = \{0\} = H \cap G_3 = H \cap G^*(\tau_i), \qquad i = 1, 2,$$

and obviously

$$H = \langle H^*(\tau_0)\rangle_* = H \cap G = H \cap G^*(\tau_0).$$

Thus H is $*$-pure in G.

Proposition 12 in [**2**] asserts that, if A is a torsion-free almost completely decomposable group, any balanced exact sequence

$$0 \to B \to C \to A \to 0$$

with C completely decomposable is $*$-balanced. A special case of the following theorem implies that B is also almost completely decomposable.

THEOREM 2. *$*$-pure subgroups of finite rank completely decomposable groups are almost completely decomposable.*

PROOF. Let H be $*$-pure in a completely decomposable group G and define

$$\mathcal{E}(H) = \{\tau \in T(H) \colon \langle H^*(\tau)\rangle_* \subsetneqq H(\tau)\},$$

where $T(H)$ is the typeset of H. Then, for every $\tau \in \mathcal{E}(H)$, $H(\tau) = H_\tau \oplus \langle H^*(\tau)\rangle_*$ where H_τ is a nonzero homogeneous completely decomposable group of type τ. If $\tau \in \mathcal{E}(H) \backslash \mathcal{E}(G)$, then $G^*(\tau) = G(\tau)$ which implies that $\langle H^*(\tau)\rangle_* = H \cap G^*(\tau) = H \cap G(\tau) = H(\tau)$, a contradiction. Thus $\mathcal{E}(H) \subseteq \mathcal{E}(G)$ and for every $0 \neq h \in H_\tau$, $\tau \in \mathcal{E}(H)$, $\pi_\tau(h) \neq 0$.

Let $\tau \in \mathcal{E}(H)$ and let $0 \neq h \in H_\tau$. We shall now prove that there exists a homomorphism $\psi \colon G \to H_\tau$ such that $\psi(h) = nh$ for some nonzero integer n. Let $\{p_1, p_2, \ldots, p_k\}$ be the set of all primes such that

$$h^G_{p_i}(h) = h^H_{p_i}(h) \lneq h^G_{p_i}(\pi_\tau(h)), \qquad i = 1, 2, \ldots, k.$$

Let $n = \prod_{i=1}^k p_i^{n(i)}$, where $n(i) = h^G_{p_i}(\pi_\tau(h) - h^G_{p_i}(h))$, $i = 1, 2, \ldots, k$.

It is easy to see that n divides $\pi_\tau(h)$ in $\langle \pi_\tau(h)\rangle_*$ and this group is a direct summand of $G_\tau = \pi_\tau(G)$. We let $\pi_\tau(h) = ng'$ and let $\theta \colon G \to \langle \pi_\tau(h)\rangle_*$ be a projection onto $\langle \pi_\tau(h)\rangle_*$. Then, for every prime $p \notin \{p_1, p_2, \ldots, p_k\}$,

$$h^H_p(h) = h^G_p(\pi_\tau(h)) = h^G_p(g')$$

and, for every $p \in \{p_1, p_2, \ldots, p_k\}$,

$$h^G_p(\pi_\tau(h)) = h^G_p(g') + h^G_p(\pi_\tau(h)) - h^G_p(h)$$

which implies that $h^G_p(g') = h^G_p(h)$. Thus, the height-sequence of g' in G is identical to the height-sequence of h in G which implies that there is an isomorphism $\psi' \colon \langle g'\rangle_* \to \langle h\rangle_*$ with $\psi'(g') = h$. We then have

$$\psi'\theta\pi_\tau(h) = \psi'\theta(ng') = n\psi'(g') = nh \quad \text{and} \quad \psi = \psi'\theta\pi_\tau \in \operatorname{Hom}(G, H_\tau).$$

We now prove that the regulating subgroup $\sum_{\tau\in\mathcal{E}(H)} H_\tau$ of H is completely decomposable. Let $0 = \sum_{i=1}^{k} h_{\tau_i}$, where $h_{\tau_i} \in H_{\tau_i}$, $\tau_i \in \mathcal{E}(H)$, $1 \le i \le k$. There exist homomorphisms $\psi_i\colon G \to H_{\tau_i}$ such that $\psi_i(h_{\tau_i}) = n_i h_{\tau_i}$ for some nonzero integers n_i, $1 \le i \le k$. If τ_1, say, is minimal in $\{\tau_1, \tau_2, \dots, \tau_k\}$ we have

$$0 = \sum_{i=1}^{k} \psi_1(h_{\tau_i}) = \psi_1(h_{\tau_1}) = n_1 h_{\tau_1}$$

which implies that $h_{\tau_1} = 0$. It is now easy to see that $h_{\tau_i} = 0$, $1 \le i \le k$, which implies that $\sum_{\tau\in\mathcal{E}(H)} H_\tau = \bigoplus_{\tau\in\mathcal{E}(H)} H_\tau$. Thus H is almost completely decomposable.

Recall that a subgroup B of a torsion-free group A is said to be *regular* in A if, for every $b \in B$, $\operatorname{type}_B(b) = \operatorname{type}_A(b)$. We shall call a subgroup B of a torsion-free group A *strongly regular* if, for every $b \in B$, there exists a nonzero integer n and a homomorphism $\psi\colon A \to B$ such that $\psi(b) = nb$. This definition is an obvious generalization of the concept of strong purity introduced by K. M. Rangaswamy and S. Janakiraman. Strongly regular subgroups are regular but not necessarily pure. A subgroup that is both strongly regular and pure will be called *strongly regular pure*. Obviously, strongly regular pure subgroups are $*$-pure but the converse is not true. For finite rank completely decomposable groups we have the following theorem.

THEOREM 3. *A subgroup H of a finite rank completely decomposable group G is $*$-pure if and only if it is strongly regular pure.*

PROOF. We need only prove that, if H is $*$-pure in G, then H is strongly regular in G. Let $\mathcal{E}(H)$ be defined as in Theorem 2. Then, for $\tau \in \mathcal{E}(H)$, $H(\tau) = H_\tau \oplus \langle H^*(\tau)\rangle_*$ and, from the proof of Theorem 2, H_τ is strongly regular in G. We shall prove that $\bigoplus_{\tau\in\mathcal{E}(G)} H_\tau$ is strongly regular in G.

Let $0 \ne h = \sum_{i=1}^{n} h_{\tau_i}$, where $0 \ne h_{\tau_i} \in H_{\tau_i}$, $\tau_i \in \mathcal{E}(H)$, $1 \le i \le n$. There exist homomorphisms $\psi_i\colon G \to \langle h_{\tau_i}\rangle_*$ and nonzero integers n_i such that $\psi_i(h_{\tau_i}) = n_i h_{\tau_i}$, $1 \le i \le n$. Then, for every $i \in \{1, 2, \dots, n\}$

$$\psi_i(h) = n_i h_{\tau_i} + \sum_{j\ne i} \psi_i(h_{\tau_j})$$

and there exist integers m_i and m_i' such that $m_i h_{\tau_i} = m_i'(\sum_{j\ne i} \psi_i(h_{\tau_j}))$, since $\sum_{j\ne i} \psi_i(h_{\tau_j}) \in \langle h_{\tau_i}\rangle_*$. Thus

$$m_i'\psi_i(h) = m_i' n_i h_{\tau_i} + m_i'\left(\sum_{j\ne i} \psi_i(h_{\tau_j})\right) = r_i h_{\tau_i}, \quad \text{where } r_i = m_i' n_i + m_i.$$

Thus

$$\begin{aligned}\left(\prod_{i=1}^{n} r_i\right) h &= \sum_{i=1}^{n}\left(\prod_{i=1}^{n} r_i\right) h_{\tau_i} = \sum_{i=1}^{n}\left(\prod_{j\neq i} r_j\right) r_i h_{\tau_i} \\ &= \sum_{i=1}^{n} r_i' m_i' \psi_i(h), \quad \text{where } r_i' = \prod_{j\neq i} r_j \\ &= \left(\sum_{i=1}^{n} r_i' m_i' \psi_i\right)(h).\end{aligned}$$

Hence

$$\psi = \sum_{i=1}^{n} r_i' m_i' \psi_i \in \operatorname{Hom}\left(G, \bigoplus_{\tau\in\mathcal{E}(H)} H_\tau\right),$$

and $\psi(h) = rh$, where $r = \prod_{i=1}^{n} r_i$. This implies that $\bigoplus_{\tau\in\mathcal{E}(H)} H_\tau$ is strongly regular in G. Since $H/\bigoplus_{\tau\in\mathcal{E}(H)} H_\tau$ is finite, for every $h' \in H$ there exists an integer n' such that $n'h' \in \bigoplus_{\tau\in\mathcal{E}(H)} H_\tau$ and from the previous paragraphs there exists $\psi\colon G \to \bigoplus_{\tau\in\mathcal{E}(H)} H_\tau \subseteq H$ such that $\psi(n'h') = rn'h' = n'\psi(h')$ for some nonzero integer r. By torsion-freeness, $\psi(h') = rh'$ which implies that H is strongly regular in G and the proof is complete.

From the proof of the previous theorem we conclude that

(i) finite rank $*$-pure subgroups of separable torsion-free groups are strongly regular,

(ii) completely decomposable $*$-pure subgroups of completely decomposable groups are strongly regular.

It seems likely that the concepts of $*$-purity and strongly regular purity are equivalent for separable torsion-free groups.

Finally, we wish to characterize those completely decomposable groups whose typesets satisfy the maximum condition any $*$-pure subgroup of which is completely decomposable.

A set of types $\mathcal{T}$ is called a *tree* if, for any two incomparable types τ_1 and τ_2 in $\mathcal{T}$, there is no τ in $\mathcal{T}$ satisfying $\tau \geq \sup\{\tau_1, \tau_2\}$.

THEOREM 4. *Let G be a reduced completely decomposable group whose typeset satisfies the maximum condition. Every $*$-pure subgroup of G is completely decomposable if and only if $\mathcal{E}(G)$ is a tree.*

PROOF. Suppose every $*$-pure subgroup of G is completely decomposable. Let $\tau_1, \tau_2 \in \mathcal{E}(G)$ be incomparable. If there exists $\tau \in \mathcal{E}(G)$ such that $\tau \gneq \tau_i$, $i = 1, 2$, then by the example after Theorem 1, G contains a $*$-pure subgroup which is not completely decomposable, a contradiction. This implies that $\mathcal{E}(G)$ is a tree.

Conversely, assume that $\mathcal{E}(G)$ is a tree and let H be a $*$-pure subgroup of G. Define

$$\mathcal{E}(H) = \{\tau \in T(H)\colon \langle H^*(\tau)\rangle_* \subsetneq H(\tau)\}.$$

Then $\mathcal{E}(H)$ is nonempty since every maximal element of $T(H)$ belongs to $\mathcal{E}(H)$. For every $\tau \in \mathcal{E}(H)$

$$\begin{aligned} H(\tau)/\langle H^*(\tau)\rangle_* &= (H \cap G(\tau))/(H \cap G^*(\tau)) \\ &\cong \{(H \cap G(\tau)) + G^*(\tau)\}/G^*(\tau) \\ &\subseteq G(\tau)/G^*(\tau) \cong G_\tau. \end{aligned}$$

Thus $H(\tau)/\langle H^*(\tau)\rangle_*$ is a homogeneous group of type τ isomorphic to a subgroup of G_τ and therefore completely decomposable. This also implies that $\langle H^*(\tau)\rangle_*$ is balanced in $H(\tau)$ and therefore $H(\tau) = H_\tau \oplus \langle H^*(\tau)\rangle_*$ where H_τ is a homogeneous completely decomposable group of type τ. From the proof of Theorem 2, H_τ is strongly regular in G for every $\tau \in \mathcal{E}(H)$ and $\sum_{\tau\in\mathcal{E}(H)} H_\tau = \bigoplus_{\tau\in\mathcal{E}(H)} H_\tau$, a completely decomposable group. We shall prove that $H = \bigoplus_{\tau\in\mathcal{E}(H)} H_\tau$.

Let $h \in H$ be of type τ_0 and, since $T(H)$ satisfies the maximum condition, we may assume that every element of H of type greater than τ_0 belongs to $\bigoplus_{\tau\in\mathcal{E}(H)} H_\tau$, i.e. $H^*(\tau_0) \subseteq \bigoplus_{\tau\in\mathcal{E}(H)} H_\tau$. We first show that this assumption implies that $\langle H^*(\tau_0)\rangle_* \subseteq \bigoplus_{\tau\in\mathcal{E}(H)} H_\tau$. Let $h' \in \langle H^*(\tau_0)\rangle_*$ and let

$$\mathcal{E}(h') = \{\tau \in \mathcal{E}(G) \colon \pi_\tau(h) \neq 0\}.$$

Since $\langle H^*(\tau_0)\rangle_* = H \cap G^*(\tau_0)$, $\tau \gneq \tau_0$ for every $\tau \in \mathcal{E}(h')$. Let $\mathcal{M}$ be the set of all minimal types in $\mathcal{E}(h')$ and, for every $\tau \in \mathcal{M}$, let

$$\mathcal{M}(\tau) = \{\tau' \in \mathcal{E}(h') \colon \tau' \geq \tau\}.$$

Since $\mathcal{E}(G)$ is a tree, $\{\mathcal{M}(\tau) \colon \tau \in \mathcal{M}\}$ is a partition of $\mathcal{E}(h')$ and also $\sum_{\tau\in\mathcal{M}} G(\tau) = \bigoplus_{\tau\in\mathcal{M}} G(\tau)$, a summand of G. From the proof of Theorem 3 $\bigoplus_{\tau\in\mathcal{E}(H)} H_\tau$ is strongly regular in G and, by assumption, there exists a nonzero integer n such that $nh' \in \bigoplus_{\tau\in\mathcal{E}(H)} H_\tau$. Thus, there is a homomorphism $\psi \colon G \to \bigoplus_{\tau\in\mathcal{E}(H)} H_\tau$ such that $\psi(h') = kh'$ for some nonzero integer k. Then

$$kh' = \sum_{\tau\in\mathcal{E}(h')} \psi\pi_\tau(h'),$$

where $\tau_0 \lneq \tau \leq \operatorname{type}_H(\psi\pi_\tau(h'))$ for every $\tau \in \mathcal{E}(h')$. By assumption, this implies that $\psi\pi_\tau(h') \in \bigoplus_{\tau\in\mathcal{E}(H)} H_\tau$. We then have

$$\begin{aligned} kh' &= \sum_{\tau\in\mathcal{E}(h')} \psi\pi_\tau(h') = \sum_{\tau\in\mathcal{M}} \sum_{\tau'\in\mathcal{M}(\tau)} \psi\pi_{\tau'}(h') \\ &\in \sum_{\tau\in\mathcal{M}} \langle \psi\pi_{\tau'}(h') \colon \tau' \in \mathcal{M}(\tau)\rangle_*, \end{aligned}$$

where $\langle \psi\pi_{\tau'}(h') \colon \tau' \in \mathcal{M}(\tau)\rangle_*$ is a pure subgroup of G for every $\tau \in \mathcal{M}$. Thus

$$\sum_{\tau\in\mathcal{M}} \langle \psi\pi_{\tau'}(h') \colon \tau' \in \mathcal{M}(\tau)\rangle_* = \bigoplus_{\tau\in\mathcal{M}} \langle \psi\pi_{\tau'}(h') \colon \tau' \in \mathcal{M}(\tau)\rangle_*$$

is pure in $\bigoplus_{\tau\in\mathcal{M}} G(\tau)$ and therefore pure in G. This implies that

$$h' \in \bigoplus_{\tau\in\mathcal{M}} \langle \psi\pi_{\tau'}(h') \colon \tau' \in \mathcal{M}(\tau)\rangle_* \subseteq \bigoplus_{\tau\in\mathcal{E}(H)} H_\tau$$

which proves that $\langle H^*(\tau_0)\rangle_* \subseteq \bigoplus_{\tau\in\mathcal{E}(H)} H_\tau$. But then $h \in H(\tau_0) = H_\tau \oplus \langle H^*(\tau_0)\rangle_*$ which proves that $H = \bigoplus_{\tau\in\mathcal{E}(H)} H_\tau$ is completely decomposable.

In the general case, one observes that if $\mathcal{E}(G)$ (and therefore $T(G)$) is linearly ordered, then every pure subgroup of G is $*$-pure. One can therefore characterize completely decomposable groups any $*$-pure subgroup of which is completely decomposable by combining Theorem 2 in [**1**] and Theorem 4 above.

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