*-PURE SUBGROUPS OF COMPLETELY DECOMPOSABLE ABELIAN GROUPS
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ABSTRACT. In this note we prove that

(i) homogeneous *-pure subgroups of completely decomposable groups are completely decomposable,

(ii) *-pure subgroups of finite rank completely decomposable groups are almost completely decomposable.

We also characterize those reduced completely decomposable groups $G$, with $T(G)$ satisfying the maximum condition, any *-pure subgroup of which is also completely decomposable.

In [2], a subgroup $H$ of a torsion-free group $G$ is said to be *-balanced if $H$ is balanced in $G$ and

$$
\langle H^*(\tau) \rangle_* = H \cap \langle G^*(\tau) \rangle_*
$$

for every type $\tau$. We shall say that $H$ is *-pure in $G$ if $H$ is pure in $G$ and equation (i) above is satisfied for every type $\tau$.

Throughout this note, unless otherwise specified, $G$ will denote a completely decomposable abelian group and its extractable typeset will be denoted by $\mathcal{E}(G)$. We let $G = \bigoplus_{\tau \in \mathcal{E}(G)} G_\tau$ be a homogeneous decomposition of $G$ and, for every $\tau \in \mathcal{E}(G)$, $\pi_\tau : G \to G_\tau$ is the projection such that $\ker \pi_\tau = \bigoplus_{\tau' \neq \tau} G_{\tau'}$.

In [4] we proved that homogeneous pure subgroups of $G$ are also completely decomposable provided that $\mathcal{E}(G)$ is countable. The following theorem implies that the countability condition of $\mathcal{E}(G)$ can be relaxed if we consider homogeneous *-pure subgroups of $G$.

THEOREM 1. Homogeneous *-pure subgroups of completely decomposable groups are completely decomposable.

PROOF. Let $H$ be a homogeneous *-pure subgroup of a completely decomposable group $G$. Let $0 \neq h \in H$ and suppose that $\text{type}_G(h) = \tau_0$. Then $h = \sum_{\tau \geq \tau_0} \pi_\tau(h)$. If $\tau_0 \notin \mathcal{E}(G)$, then

$$
h = \sum_{\tau > \tau_0} \pi_\tau(h) \in H \cap G^*(\tau_0) = \langle H^*(\tau_0) \rangle_* = \{0\},
$$

a contradiction. Thus $\tau_0 \in \mathcal{E}(G)$ and $\pi_{\tau_0}(h) \neq 0$. Hence $H \cong \pi_{\tau_0}(H) \subseteq G_{\tau_0}$. Since $\pi_{\tau_0}(H)$ is homogeneous of type $\tau_0$ and is a subgroup of a homogeneous completely decomposable group $G_{\tau_0}$, also of type $\tau_0$, it is completely decomposable by Lemma 86.6 in [3].

The following example illustrates that the homogeneity condition on $H$ in the previous theorem is necessary.
EXAMPLE. Let $G = G_1 \oplus G_2 \oplus G_3$ be a torsion-free group, where $G_i$ is of rank one and type $\tau_i$ and $G_i \not\cong Q$, the group of rationals, $1 \leq i \leq 3$. Assume that $\tau_1$ and $\tau_2$ are incomparable and $\tau_3 \geq \sup\{\tau_1, \tau_2\} > \tau_i$, $i = 1, 2$. By Lemma 1 in [1], $G$ contains an indecomposable pure subgroup $H$ with typeset $T(H) = \{\tau_0, \tau_1, \tau_2\}$, where $\tau_0 = \inf\{\tau_1, \tau_2\}$. Since $\tau_i$ is maximal in $T(H)$,

$$\langle H^*(\tau_i) \rangle \ast = \{0\} = H \cap G_3 = H \cap G^*(\tau_i), \quad i = 1, 2,$$

and obviously

$$H = \langle H^*(\tau_0) \rangle \ast = H \cap G = H \cap G^*(\tau_0).$$

Thus $H$ is $\ast$-pure in $G$.

Proposition 12 in [2] asserts that, if $A$ is a torsion-free almost completely decomposable group, any balanced exact sequence

$$0 \to B \to C \to A \to 0$$

with $C$ completely decomposable is $\ast$-balanced. A special case of the following theorem implies that $B$ is also almost completely decomposable.

THEOREM 2. $\ast$-pure subgroups of finite rank completely decomposable groups are almost completely decomposable.

PROOF. Let $H$ be $\ast$-pure in a completely decomposable group $G$ and define

$$E(H) = \{\tau \in T(H) : \langle H^*(\tau) \rangle \ast \subsetneq H(\tau)\},$$

where $T(H)$ is the typeset of $H$. Then, for every $\tau \in E(H)$, $H(\tau) = H_\tau \oplus \langle H^*(\tau) \rangle \ast$ where $H_\tau$ is a nonzero homogeneous completely decomposable group of type $\tau$. If $\tau \in E(H) \setminus E(G)$, then $G^*(\tau) = G(\tau)$ which implies that $\langle H^*(\tau) \rangle \ast = H \cap G^*(\tau) = H \cap G(\tau) = H(\tau)$, a contradiction. Thus $E(H) \subseteq E(G)$ and for every $0 \neq h \in H_\tau$, $\tau \in E(H)$, $H(\tau) \not\cong 0$.

Let $\tau \in E(H)$ and let $0 \neq h \in H_\tau$. We shall now prove that there exists a homomorphism $\psi : G \to H_\tau$ such that $\psi(h) = nh$ for some nonzero integer $n$. Let $\{p_1, p_2, \ldots, p_k\}$ be the set of all primes such that

$$h_{p_i}(h) = h_{p_i}^G(h) \leq h_{p_i}^G(\pi_\tau(h)), \quad i = 1, 2, \ldots, k.$$

Let $n = \prod_{i=1}^k p_i^{n(i)}$, where $n(i) = h_{p_i}^G(\pi_\tau(h)) - h_{p_i}^G(h)$, $i = 1, 2, \ldots, k$.

It is easy to see that $n$ divides $\pi_\tau(h)$ in $\langle \pi_\tau(h) \rangle \ast$ and this group is a direct summand of $G_\tau = \pi_\tau(G)$. We let $\pi_\tau(h) = ng'$ and let $\theta : G \to \langle \pi_\tau(h) \rangle \ast$ be a projection onto $\langle \pi_\tau(h) \rangle \ast$. Then, for every prime $p \not\in \{p_1, p_2, \ldots, p_k\}$,

$$h_p^G(h) = h_p^G(\pi_\tau(h)) = h_p^G(g')$$

and, for every $p \in \{p_1, p_2, \ldots, p_k\}$,

$$h_p^G(\pi_\tau(h)) = h_p^G(g') + h_p^G(\pi_\tau(h)) - h_p^G(h)$$

which implies that $h_p^G(g') = h_p^G(h)$. Thus, the height-sequence of $g'$ in $G$ is identical to the height-sequence of $h$ in $G$ which implies that there is an isomorphism $\psi' : \langle g' \rangle \ast \to \langle h \rangle \ast$ with $\psi'(g') = h$. We then have

$$\psi' \pi_\tau(h) = \psi'(ng') = n\psi'(g') = nh \quad \text{and} \quad \psi = \psi' \theta \pi_\tau \in \text{Hom}(G, H_\tau).$$
We now prove that the regulating subgroup $\sum_{\tau \in \mathcal{E}(H)} H_{\tau}$ of $H$ is completely decomposable. Let $0 = \sum_{i=1}^{k} h_{\tau_i}$, where $h_{\tau_i} \in H_{\tau_i}$, $\tau_i \in \mathcal{E}(H)$, $1 \leq i \leq k$. There exist homomorphisms $\psi_i : G \to H_{\tau_i}$ such that $\psi_i(h_{\tau_i}) = n_i h_{\tau_i}$ for some nonzero integers $n_i$, $1 \leq i \leq k$. If $\tau_1$, say, is minimal in $\{\tau_1, \tau_2, \ldots, \tau_k\}$ we have

$$0 = \sum_{i=1}^{k} \psi_i(h_{\tau_i}) = \psi_1(h_{\tau_1}) = n_1 h_{\tau_1}$$

which implies that $h_{\tau_1} = 0$. It is now easy to see that $h_{\tau_i} = 0$, $1 \leq i \leq k$, which implies that $\sum_{\tau \in \mathcal{E}(H)} H_{\tau} = \bigoplus_{\tau \in \mathcal{E}(H)} H_{\tau}$. Thus $H$ is almost completely decomposable.

Recall that a subgroup $B$ of a torsion-free group $A$ is said to be regular in $A$ if, for every $b \in B$, $\text{type}_B(b) = \text{type}_A(b)$. We shall call a subgroup $B$ of a torsion-free group $A$ strongly regular if, for every $b \in B$, there exists a nonzero integer $n$ and a homomorphism $\psi : A \to B$ such that $\psi(b) = nb$. This definition is an obvious generalization of the concept of strong purity introduced by K. M. Rangaswamy and S. Janakiraman. Strongly regular subgroups are regular but not necessarily pure. A subgroup that is both strongly regular and pure will be called strongly regular pure. Obviously, strongly regular pure subgroups are $*$-pure but the converse is not true. For finite rank completely decomposable groups we have the following theorem.

**Theorem 3.** A subgroup $H$ of a finite rank completely decomposable group $G$ is $*$-pure if and only if it is strongly regular pure.

**Proof.** We need only prove that, if $H$ is $*$-pure in $G$, then $H$ is strongly regular in $G$. Let $\mathcal{E}(H)$ be defined as in Theorem 2. Then, for $\tau \in \mathcal{E}(H)$, $H(\tau) = H_{\tau} \oplus \langle H^*(\tau) \rangle_{\tau}$ and, from the proof of Theorem 2, $H_{\tau}$ is strongly regular in $G$. We shall prove that $\bigoplus_{\tau \in \mathcal{E}(G)} H_{\tau}$ is strongly regular in $G$.

Let $0 \neq h = \sum_{i=1}^{n} h_{\tau_i}$, where $0 \neq h_{\tau_i} \in H_{\tau_i}$, $\tau_i \in \mathcal{E}(H)$, $1 \leq i \leq n$. There exist homomorphisms $\psi_i : G \to \langle h_{\tau_i} \rangle_{\tau}$ and nonzero integers $n_i$ such that $\psi_i(h_{\tau_i}) = n_i h_{\tau_i}$, $1 \leq i \leq n$. Then, for every $i \in \{1, 2, \ldots, n\}$

$$\psi_i(h) = n_i h_{\tau_i} + \sum_{j \neq i} \psi_j(h_{\tau_j})$$

and there exist integers $m_i$ and $m'_i$ such that $m_i h_{\tau_i} = m'_i (\sum_{j \neq i} \psi_j(h_{\tau_j}))$, since $\sum_{j \neq i} \psi_j(h_{\tau_j}) \in \langle h_{\tau_i} \rangle_{\tau}$. Thus

$$m'_i \psi_i(h) = m'_i n_i h_{\tau_i} + m'_i \left( \sum_{j \neq i} \psi_j(h_{\tau_j}) \right) = r_i h_{\tau_i},$$

where $r_i = m'_i n_i + m_i$. 


Thus
\[
\left( \prod_{i=1}^{n} r_i \right) h = \sum_{i=1}^{n} \left( \prod_{i=1}^{n} r_i \right) h_{r_i} = \sum_{i=1}^{n} \left( \prod_{j \neq i} r_j \right) r_i h_{r_i} \\
= \sum_{i=1}^{n} r'_i m'_i \psi_i(h), \quad \text{where } r'_i = \prod_{j \neq i} r_j \\
= \left( \sum_{i=1}^{n} r'_i m'_i \psi_i \right) (h).
\]

Hence
\[
\psi = \sum_{i=1}^{n} r'_i m'_i \psi_i \in \text{Hom} \left( G, \bigoplus_{\tau \in \mathcal{E}(H)} H_{\tau} \right),
\]
and \( \psi(h) = rh \), where \( r = \prod_{i=1}^{n} r_i \). This implies that \( \bigoplus_{\tau \in \mathcal{E}(H)} H_{\tau} \) is strongly regular in \( G \). Since \( H/\bigoplus_{\tau \in \mathcal{E}(H)} H_{\tau} \) is finite, for every \( h' \in H \) there exists an integer \( n' \) such that \( n'h' \in \bigoplus_{\tau \in \mathcal{E}(H)} H_{\tau} \) and from the previous paragraphs there exists \( \psi: G \to \bigoplus_{\tau \in \mathcal{E}(H)} H_{\tau} \subseteq H \) such that \( \psi(n'h') = r n'h' = n' \psi(h') \) for some nonzero integer \( r \). By torsion-freeness, \( \psi(h') = rh' \) which implies that \( H \) is strongly regular in \( G \) and the proof is complete.

From the proof of the previous theorem we conclude that
(i) finite rank \(*\)-pure subgroups of separable torsion-free groups are strongly regular,
(ii) completely decomposable \(*\)-pure subgroups of completely decomposable groups are strongly regular.

It seems likely that the concepts of \(*\)-purity and strongly regular purity are equivalent for separable torsion-free groups.

Finally, we wish to characterize those completely decomposable groups whose type sets satisfy the maximum condition any \(*\)-pure subgroup of which is completely decomposable.

A set of types \( T \) is called a tree if, for any two incomparable types \( \tau_1 \) and \( \tau_2 \) in \( T \), there is no \( \tau \) in \( T \) satisfying \( \tau \supseteq \sup \{ \tau_1, \tau_2 \} \).

**Theorem 4.** Let \( G \) be a reduced completely decomposable group whose type set satisfies the maximum condition. Every \(*\)-pure subgroup of \( G \) is completely decomposable if and only if \( \mathcal{E}(G) \) is a tree.

**Proof.** Suppose every \(*\)-pure subgroup of \( G \) is completely decomposable. Let \( \tau_1, \tau_2 \in \mathcal{E}(G) \) be incomparable. If there exists \( \tau \in \mathcal{E}(G) \) such that \( \tau \supseteq \tau_i, i = 1, 2 \), then by the example after Theorem 1, \( G \) contains a \(*\)-pure subgroup which is not completely decomposable, a contradiction. This implies that \( \mathcal{E}(G) \) is a tree.

Conversely, assume that \( \mathcal{E}(G) \) is a tree and let \( H \) be a \(*\)-pure subgroup of \( G \). Define
\[
\mathcal{E}(H) = \{ \tau \in T(H): \langle H^*(\tau) \rangle_* \not\subseteq H(\tau) \}.
\]
Then $\mathcal{E}(H)$ is nonempty since every maximal element of $T(H)$ belongs to $\mathcal{E}(H)$. For every $\tau \in \mathcal{E}(H)$,

$$H(\tau)/\langle H^*(\tau) \rangle_\ast = (H \cap G(\tau))/(H \cap G^*(\tau))$$

$$\cong \{(H \cap G(\tau)) + G^*(\tau)\}/G^*(\tau)$$

$$\subseteq G(\tau)/G^*(\tau) \cong G_\tau.$$ 

Thus $H(\tau)/\langle H^*(\tau) \rangle_\ast$ is a homogeneous group of type $\tau$ isomorphic to a subgroup of $G_\tau$ and therefore completely decomposable. This also implies that $\langle H^*(\tau) \rangle_\ast$ is balanced in $H(\tau)$ and therefore $H(\tau) = H_\tau \oplus \langle H^*(\tau) \rangle_\ast$ where $H_\tau$ is a homogeneous completely decomposable group of type $\tau$. From the proof of Theorem 2, $H_\tau$ is strongly regular in $G$ for every $\tau \in \mathcal{E}(H)$ and $\sum_{\tau \in \mathcal{E}(H)} H_\tau = \bigoplus_{\tau \in \mathcal{E}(H)} H_\tau$, a completely decomposable group. We shall prove that $H = \bigoplus_{\tau \in \mathcal{E}(H)} H_\tau$.

Let $h \in H$ be of type $\tau_0$ and, since $T(H)$ satisfies the maximum condition, we may assume that every element of $H$ of type greater than $\tau_0$ belongs to $\bigoplus_{\tau \in \mathcal{E}(H)} H_\tau$, i.e. $H^*(\tau_0) \subseteq \bigoplus_{\tau \in \mathcal{E}(H)} H_\tau$. We first show that this assumption implies that $\langle H^*(\tau_0) \rangle_\ast \subseteq \bigoplus_{\tau \in \mathcal{E}(H)} H_\tau$. Let $h' \in \langle H^*(\tau_0) \rangle_\ast$ and let

$$\mathcal{E}(h') = \{\tau \in \mathcal{E}(G): \pi_\tau(h) \neq 0\}.$$ 

Since $\langle H^*(\tau_0) \rangle_\ast = H \cap G^*(\tau_0)$, $\tau \geq \tau_0$ for every $\tau \in \mathcal{E}(h')$. Let $M$ be the set of all minimal types in $\mathcal{E}(h')$ and, for every $\tau \in M$, let

$$M(\tau) = \{\tau' \in \mathcal{E}(h'): \tau' \geq \tau\}.$$ 

Since $\mathcal{E}(G)$ is a tree, $\{M(\tau): \tau \in M\}$ is a partition of $\mathcal{E}(h')$ and also $\sum_{\tau \in M} G(\tau) = \bigoplus_{\tau \in M} G(\tau)$, a summand of $G$. From the proof of Theorem 3 $\bigoplus_{\tau \in \mathcal{E}(H)} H_\tau$ is strongly regular in $G$ and, by assumption, there exists a nonzero integer $n$ such that $nh' \in \bigoplus_{\tau \in \mathcal{E}(H)} H_\tau$. Thus, there is a homomorphism $\psi: G \to \bigoplus_{\tau \in \mathcal{E}(H)} H_\tau$ such that $\psi(h') = kh'$ for some nonzero integer $k$. Then

$$kh' = \sum_{\tau \in \mathcal{E}(h')} \psi_{\pi_\tau}(h'),$$ 

where $\tau_0 \leq \tau \leq \text{type}_H(\psi_{\pi_\tau}(h'))$ for every $\tau \in \mathcal{E}(h')$. By assumption, this implies that $\psi_{\pi_\tau}(h') \in \bigoplus_{\tau \in \mathcal{E}(H)} H_\tau$. We then have

$$kh' = \sum_{\tau \in \mathcal{E}(h')} \psi_{\pi_\tau}(h') = \sum_{\tau \in M} \sum_{\tau' \in M(\tau)} \psi_{\pi_{\tau'}}(h')$$

$$\in \sum_{\tau \in \mathcal{M}} \langle \psi_{\pi_{\tau'}}(h'): \tau' \in M(\tau) \rangle_\ast,$$

where $\langle \psi_{\pi_{\tau'}}(h'): \tau' \in M(\tau) \rangle_\ast$ is a pure subgroup of $G$ for every $\tau \in \mathcal{M}$. Thus

$$\sum_{\tau \in \mathcal{M}} \langle \psi_{\pi_{\tau'}}(h'): \tau' \in M(\tau) \rangle_\ast = \bigoplus_{\tau \in \mathcal{M}} \langle \psi_{\pi_{\tau'}}(h'): \tau' \in M(\tau) \rangle_\ast$$

is pure in $\bigoplus_{\tau \in \mathcal{M}} G(\tau)$ and therefore pure in $G$. This implies that

$$h' \in \bigoplus_{\tau \in \mathcal{M}} \langle \psi_{\pi_{\tau'}}(h'): \tau' \in M(\tau) \rangle_\ast \subseteq \bigoplus_{\tau \in \mathcal{E}(H)} H_\tau.$$
which proves that \( (H^*(\sigma_0))^* \subseteq \bigoplus_{r \in \mathcal{E}(H)} H_r \). But then \( h \in H(\sigma_0) = H_r \oplus (H^*(\sigma_0))^* \) which proves that \( H = \bigoplus_{r \in \mathcal{E}(H)} H_r \) is completely decomposable.

In the general case, one observes that if \( \mathcal{E}(G) \) (and therefore \( T(G) \)) is linearly ordered, then every pure subgroup of \( G \) is \(*\)-pure. One can therefore characterize completely decomposable groups any \(*\)-pure subgroup of which is completely decomposable by combining Theorem 2 in [1] and Theorem 4 above.

REFERENCES