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ON THE PURELY INSEPARABLE CLOSURE OF RINGS

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ABSTRACT. Let $K \subseteqq R$ be commutative rings with identity 1. Let $D = \{D_i\}$ be a higher derivation of R. We shall prove in this paper that if K is invariant with respect to D, the purely inseparable closure $\overline{K_R}$ of K in R is invariant with respect to D and the formal power series ring $\overline{K_R}[[t]]$ is purely inseparably closed in $R[[t]]$.

1. Introduction. Let $K \subseteqq R$ be commutative rings with identity 1. Let $D = \{D_i\}$ be a higher derivation of R. We shall write $D(K) \subseteqq K$ if $D_i(K) \subseteqq K$ for all i. We shall call R purely inseparable over K if for every element x of R there is an integer n such that $(x \otimes 1 - 1 \otimes x)^n = 0$ in $R \otimes_K R$ (cf. [**1**]). It is immediately proved that there exists the maximum purely inseparable subalgebra over K contained in R. We shall call this subalgebra the purely inseparable closure of K in R and we shall write $\overline{K_R}$. We shall prove that $D(K) \subseteq K$ implies $D(\overline{K_R}) \subseteqq \overline{K_R}$ and the formal power series ring $\overline{K_R}[[t]]$ is purely inseparably closed in $R[[t]]$.

2. The purely inseparable closure. Let $K \subseteqq R$ be commutative rings with identity 1. Let $\{K_\lambda\}_{\lambda \in \Lambda}$ be the set of purely inseparable subalgebras over K contained in R and let $\overline{K_R}$ be the subalgebra of R generated by $\{K_\lambda\}_{\lambda \in \Lambda}$ over K. Then it is evident that $\overline{K_R}$ is the maximum purely inseparable subalgebra over K in R. We shall call $\overline{K_R}$ the purely inseparable closure of K in R. If $\overline{K_R} = K$, we shall call K purely inseparably closed in R.

EXAMPLE. Let K be a field of characteristic $p > 0$ and let X, Y, and Z be three indeterminates over K. Let $R = K[X, Y, Z]/(Y^p - X) = K[x, y, z]$. Then R is a domain and $K[x, y]$ is purely inseparable over $K[x]$. Assume S is a purely inseparable subalgebra over $K[x]$ containing $K[x, y]$. An element $f \in S$ can be written in the form

$$f = \sum_{i=0}^{n} g_i(x, y) z^i, \qquad g_i(x, y) \in K[x, y].$$

Then we have $f^{p^m} \otimes 1 - 1 \otimes f^{p^m} = 0$ for some integer m in $S \otimes_{K[x]} S$. Since $g_i(x, y)^{p^m} \in K[x]$, we have

$$\sum_{i=1}^{n} g_i(x, y)^{p^m} (z^{ip^m} \otimes 1 - 1 \otimes z^{ip^m}) = 0$$

and hence

$$g_i(x, y)^{p^m} = 0 \quad \text{for all } i \geq 1.$$

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Since R is a domain D, we have $g_i(x,y)=0$ for all i and $f=g_0(x,y)$. Therefore we have $S \subseteqq K[x,y]$ and it follows that the purely inseparable closure $\overline{K[x]_R}$ of $K[x]$ in R is $K[x,y]$.

A higher derivation D of R is an infinite sequence $D=\{D_0,D_1,D_2,\dots\}$ of mappings D_i of R into R such that

$$D_i(x+y)=D_i(x)+D_i(y),$$

$$D_i(xy)=\sum_{j=0}^{i} D_j(x)D_{i-j}(y) \quad \text{for } x,y\in R,$$

$$D_0(1)=1.$$

Let $D=\{D_i\}$ be a higher derivation of R and let $e^{tD}=D_0+tD_1+t^2D_2+\cdots$ (t is a variable). Then e^{tD} is a homomorphism of R into $R[[t]]$.

LEMMA 1. *Let $K \subseteqq S \subseteqq R$ be commutative rings with identity* 1. *Let $D=\{D_i\}$ be a higher derivation of R. Assume S is a subalgebra of R purely inseparable over K. If $D(K) \subseteqq K$, then $e^{tD}(S)$ is a purely inseparable subalgebra of $R[[t]]$ over $e^{tD}(K)$.*

PROOF. Let L be a reduced $e^{tD}(K)$-algebra and let ψ,ϕ be $e^{tD}(K)$-algebra homomorphisms of $e^{tD}(S)$ into L such that $\psi i=\phi i$ where i is the canonical injection of $e^{tD}(K)$ into $e^{tD}(S)$. We shall consider the commutative diagram:

$$\begin{tikzcd}
R \arrow[r] & R[[t]] & & \\
S \arrow[u] \arrow[r, "e^{tD}"] & e^{tD}(S) \arrow[u] \arrow[r, shift left, "\psi"] \arrow[r, shift right, "\phi"'] & L & \\
K \arrow[u, "i"] \arrow[r, "e^{tD}"] & e^{tD}(K) \arrow[u, "i"] \arrow[r] & 0 &
\end{tikzcd}$$

Then, since L is regarded as a K-algebra by e^{tD}, we have $\psi i e^{tD}=\phi i e^{tD}$. Hence we have $\psi i e^{tD}=\psi e^{tD} i$ and $\phi i e^{tD}=\phi e^{tD} i$. Since S is purely inseparable over K, it holds that $\psi e^{tD}=\phi e^{tD}$ and hence $\psi=\phi$. Therefore $e^{tD}(S)$ is purely inseparable over $e^{tD}(K)$.

LEMMA 2. *Using the same terminology as in Lemma* 1, *let T be an intermediate ring as $K \subseteqq T \subseteqq R$. Then if S is a subalgebra of R purely inseparable over K then ST is purely inseparable over T.*

PROOF. Let U be a reduced T-algebra and let ψ,ϕ be T-algebra homomorphisms of ST into U satisfying $\psi i=\phi i$. We shall consider the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r, "j"] & ST \arrow[r, shift left, "\psi"] \arrow[r, shift right, "\phi"'] & U \\
0 \arrow[r] & K \arrow[u, "i"] \arrow[r, "j"] & T \arrow[u, "i"] &
\end{tikzcd}$$

(i,j are the canonical injections). From $\psi i=\phi i$ we have $\psi ij=\phi ij$ and hence $\psi ji=\phi ji$. Since S is purely inseparable over K, it holds that $\psi=\phi$ and therefore ST is purely inseparable over T.

LEMMA 3. *Let $K \subseteqq R$ be commutative rings with identity 1 and let f be a mapping of the product space $R[[t]] \times R[[t]]$ into $(R \otimes_K R)[[t]]$ (t is a variable) such that*

$$f\left(\sum a_i t^i, \sum b_i t^i\right) = \sum_{k=0}^{\infty} \sum_{i+j=k} (a_i \otimes b_j) t^k, \qquad a_i, b_j \in R.$$

Then f is bilinear and hence we have a linear mapping $\bar{f}$ of $R[[t]] \otimes_{K[[t]]} R[[t]]$ into $(R \otimes_K R)[[t]]$. By a simple calculation, this mapping is a ring homomorphism.

THEOREM 1. *Let $K \subseteqq R$ be commutative rings with identity 1 and let $D = \{D_i\}$ be a higher derivation of R. If $D(K) \subseteqq K$, we have $D(\overline{K_R}) \subseteqq \overline{K_R}$.*

PROOF. Let S be the subalgbra of R generated by $\{D_i(a)\}_{a \in \overline{K_R}, i}$ over $\overline{K_R}$. By Lemmas 1 and 2, $e^{tD}(\overline{K_R})K[[t]]$ is purely inseparable over $e^{tD}(K)K[[t]] = K[[t]]$. Therefore, for any element $a \in \overline{K_R}$, there is an integer n such that

$$(e^{tD}(a) \otimes 1 - 1 \otimes e^{tD}(a))^n = 0$$

in

$$e^{tD}(\overline{K_R})K[[t]] \ \otimes K[[t]] \ e^{tD}(\overline{K_R})K[[t]].$$

Since $e^{tD}(\overline{K_R})K[[t]] \subseteqq S[[t]]$, we have $e^{tD}(a) \otimes 1 - 1 \otimes e^{tD}(a))^n = 0$ in $S[[t]] \otimes_{K[[t]]} S[[t]]$ and hence

$$\bar{f}(e^{tD}(a) \otimes 1 - 1 \otimes e^{tD}(a))^n = 0$$

in $(S \otimes_K S)[[t]]$ by Lemma 3. By a simple calculation we have

$$\left[e^{tD}(a) \otimes 1 - 1 \otimes e^{tD}(a) - \sum_{i=0}^{m-1} (D_i(a) \otimes 1 - 1 \otimes D_i(a)) t^i\right]^{n2^m} = 0$$

in $(S \otimes_K S)[[t]]$ and hence $(D_m(a) \otimes 1 - 1 \otimes D_m(a))^{n2^m} = 0$ in $S \otimes_K S$. Since S is generated by $\{D_i(a)\}_{0 \le i < \infty, a \in \overline{K_R}}$ over $\overline{K_R}$, S is purely inseparable over K. Therefore we have $\overline{K_R} = S$ and $D(\overline{K_R}) \subseteqq \overline{K_R}$.

THEOREM 2. *Let $K \subseteqq R$ be commutative rings with identity 1. Then $\overline{K_R}[[t]]$ is purely inseparably closed in $R[[t]]$.*

PROOF. Let $D = \{D_k\}$ be a sequence of mappings of $R[[t]]$ into $R[[t]]$ such that $D_k(\sum a_i t^i) = a_k t^k$ for $\sum a_i t^i \in R[[t]]$, $k = 0, 1, 2, 3, \ldots$. Then D is a higher derivation of $R[[t]]$ such that $D(\overline{K_R}[[t]]) \subseteqq \overline{K_R}[[t]]$.

Let T be the purely inseparable closure of $\overline{K_R}[[t]]$ in $R[[t]]$ and let $\alpha = \sum a_i t^i \in T$. By Theorem 1 we have $D_k(\alpha) \in T$ and hence $(a_k \otimes 1 - 1 \otimes a_k)^n (t^{nk} \otimes 1) = 0$ in $T \otimes_{\overline{K_R}[[t]]} T$ for some integer n. Let S be a subring of R generated by coefficients of elements of T. Then we have $(a_k \otimes 1 - 1 \otimes a_k)^n (t^n \otimes 1) = 0$ in $S[[t]] \otimes_{\overline{K_R}[[t]]} S[[t]]$ and hence by Lemma 3 $(a_k \otimes 1 - 1 \otimes a_k)^n = 0$ in $S \otimes_{\overline{K_R}} S$. By the definition of S, S is purely inseparable over $\overline{K_R}$. Since $\overline{K_R}$ is purely inseparably closed in R, we have $\overline{K_R} = S$. Therefore it follows $T = \overline{K_R}[[t]]$ and hence $\overline{K_R}[[t]]$ is purely inseparably closed in $R[[t]]$.

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