INVARIANCE UNDER OPERATION A

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ABSTRACT. The invariance under operation $A$ of the families of sets having the classical Baire property, of Lebesgue measurable sets, and of Marczewski sets is established in a unified manner.

Marczewski has formulated a general theorem which simultaneously implies the invariance under the set-theoretical operation $A$ of the family of Lebesgue measurable sets and the family of sets having the classical Baire property (see [18, 19] and, for related matters, [5–8, 12–15, 18, 22]). In [21] Marczewski further established the invariance under operation $A$ of a new family of sets, which we call Marczewski sets, but utilized a different method of proof. By a suitable modification of Marczewski's general argument we unify these three results here and establish the invariance under operation $A$ for any category base.

For the relevant definitions and properties of category bases used below see [11]. For additional classifications of sets invariant under operation $A$ see [10].

THEOREM. The family of sets having the Baire property with respect to any category base is invariant under operation $A$.

PROOF. We denote by $Z$ the set of all infinite sequences $\nu = (\nu_1, \nu_2, \ldots)$ of natural numbers and by $N^k$ the set of all $k$-tuples $(\nu_1, \nu_2, \ldots, \nu_k)$ whose terms are elements of the set $N$ of natural numbers.

Let

$$S = \bigcup_{\nu \in Z} \bigcap_{k \geq 1} S_{\nu_1 \cdots \nu_k}$$

be the nucleus of a determinant system of sets $S_{\nu_1 \cdots \nu_k}$ which have the Baire property. The family of sets which have the Baire property being closed under finite intersections, we may assume, without loss of generality, that for each sequence $\nu = (\nu_1, \nu_2, \ldots) \in Z$ and each $k \in N$ we have

$$S_{\nu_1 \cdots \nu_{k+1}} \subset S_{\nu_1 \cdots \nu_k}.$$

(Otherwise, setting

$$S'_{\nu_1 \cdots \nu_k} = \bigcap_{p=1}^{k} S_{\nu_1 \cdots \nu_p}$$

for all $\nu \in Z$ and all $k \in N$, we obtain a determinant system of sets $S'_{\nu_1 \cdots \nu_k}$ having the Baire property which satisfies this inclusion and whose nucleus is also $S$.)

Received by the editors May 15, 1986.

1980 Mathematics Subject Classification (1985 Revision). Primary 28A05, 54A05; Secondary 54E52, 54H05.

Key words and phrases. Category base, Baire property, operation $A$. 

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0002-9939/87 $1.00 + .25$ per page

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In order to show that $S$ has the Baire property it suffices to show that if $A$ is any region in which $S$ is abundant everywhere, then $A - S$ is a meager set. Assume therefore that $A$ is such a region.

Suppose $k \in \mathbb{N}$ and $\alpha = (\nu_1, \ldots, \nu_k) \in \mathbb{N}^k$. Define

$$T_\alpha = \bigcup_{\mu \in \mathbb{Z}} \bigcap_{j=1}^{\infty} S_{\\nu_1 \ldots \nu_k, \mu_1 \ldots \mu_j}.$$ 

We proceed to define a particular maximal family (possibly empty) $\mathcal{M}_\alpha$ of disjoint subregions of $A$ such that $T_\alpha$ is abundant everywhere in each region in $\mathcal{M}_\alpha$.

Let $\mathcal{N}_\alpha$ consist of all those regions in which $T_\alpha$ is either meager or abundant everywhere. Then $(\bigcup \mathcal{N}_\alpha, \mathcal{N}_\alpha)$ is a category base. Applying Lemma 2 of [11], we define $\mathcal{M}_\alpha^*$ to be a subfamily of $\mathcal{N}_\alpha$, consisting of disjoint regions, having the property that for every region $N \in \mathcal{N}_\alpha$ there is a region $M \in \mathcal{M}_\alpha^*$ such that $N \cap M$ contains a region in $\mathcal{N}_\alpha$. Set

$$\mathcal{M}_\alpha = \{ M \in \mathcal{M}_\alpha^*: T_\alpha \text{ is abundant everywhere in } M \}.$$ 

Now defining $R_\alpha = S_\alpha \cap (\bigcup \mathcal{M}_\alpha)$, we have $R_\alpha \subset S_\alpha$. Set $Q = A - \bigcup_{m=1}^{\infty} R_m$ and, for each $k \in \mathbb{N}$ and $(\nu_1, \ldots, \nu_k) \in \mathbb{N}^k$, set

$$Q_{\nu_1 \ldots \nu_k} = R_{\nu_1 \ldots \nu_k} - \bigcup_{m=1}^{\infty} R_{\nu_1 \ldots \nu_k m}.$$ 

We then have

$$A - S = A - \bigcup_{\nu \in \mathbb{Z}} \bigcap_{k=1}^{\infty} S_{\nu_1 \ldots \nu_k} \subset A - \bigcup_{\nu \in \mathbb{Z}} \bigcap_{k=1}^{\infty} R_{\nu_1 \ldots \nu_k}$$

$$\subset \left( A - \bigcup_{m=1}^{\infty} R_m \right) \cup \left[ \bigcup_{\nu \in \mathbb{Z}} \bigcap_{k=1}^{\infty} \left( R_{\nu_1 \ldots \nu_k} - \bigcup_{m=1}^{\infty} R_{\nu_1 \ldots \nu_k m} \right) \right]$$

$$= Q \cup \left( \bigcup_{k=1}^{\infty} \bigcup_{\alpha} Q_\alpha \right),$$

where $\alpha$ varies over all sequences $\alpha = (\nu_1, \ldots, \nu_k) \in \mathbb{N}^k$ for each $k \in \mathbb{N}$. Now, the totality of sets $Q_\alpha$ is countable. Hence, in order to show that $A - S$ is a meager set, we have only to show $Q$ and all the sets $Q_\alpha$ are meager sets.

Suppose $Q$ is not meager. Being a subset of $A$, the set $Q$ is abundant everywhere in a region $B \subset A$. From the inclusion $S \subset \bigcup_{n=1}^{\infty} T_n$ and the fact that $S$ is abundant in $B$, it follows that there is an index $n_1$ such that $T_{n_1}$ is abundant in $B$. There is then a subregion $N$ of $B$ in which $T_{n_1}$ is abundant everywhere. According to the definition of the family $\mathcal{M}_{n_1}^*$, there exists a region $M \in \mathcal{M}_{n_1}^*$ such that $N \cap M$ contains a region $C$ in which $T_{n_1}$ is abundant everywhere. As $T_{n_1} \subset S_{n_1}$, the set $S_{n_1}$ is also abundant everywhere in $C$. Now, we have

$$S_{n_1} \cap C \subset R_{n_1} \subset \bigcup_{m=1}^{\infty} R_m,$$

which implies

$$Q \subset X - \bigcup_{m=1}^{\infty} R_m \subset X - (S_{n_1} \cap C).$$
Hence, \( Q \) being abundant everywhere in \( C \), the set \( X = (S_{n_1} \cap C) \) is abundant everywhere in \( C \). Because \( S_{n_1} \) is also abundant everywhere in \( C \) and both \( S_{n_1} \) and \( X = (S_{n_1} \cap C) \) have the Baire property, the set
\[
S_{n_1} \cap [X - (S_{n_1} \cap C)] = S_{n_1} - C
\]
is abundant in \( C \). But this is impossible! We conclude \( Q \) must be a meager set.

Suppose \( k \in \mathbb{N} \) and \( \alpha = (\nu_1, \ldots, \nu_k) \in \mathbb{N}^k \). To show that \( Q_\alpha \) is a meager set, we assume to the contrary that \( Q_\alpha \) is abundant. Then \( Q_\alpha \) is abundant everywhere is some region \( D \).

The set \( Q_\alpha \) is abundant everywhere is some region \( B \in \mathcal{N}_\alpha \). For, if \( T_\alpha \) is meager in \( D \) then \( D \in \mathcal{N}_\alpha \), so we may take \( B = D \). Whereas, if \( T_\alpha \) is abundant in \( D \) then \( T_\alpha \) is abundant everywhere is some region \( B \subset D \), so \( B \in \mathcal{N}_\alpha \) and \( Q_\alpha \) is abundant everywhere in \( B \).

The set \( T_\alpha \) must also be abundant everywhere in \( B \). For, suppose \( T_\alpha \) is meager in some region \( B' \subset B \). Then there exists a region \( M^* \in \mathcal{M}_\alpha^* \) and a region \( B'' \in \mathcal{N}_\alpha \) such that \( B'' \subset B' \cap M^* \). The set \( T_\alpha \) cannot be abundant everywhere in \( M^* \) and, consequently, \( M^* \notin \mathcal{M}_\alpha \). The regions in \( \mathcal{M}_\alpha^* \) being disjoint, we have \( M^* \cap (\bigcup \mathcal{M}_\alpha) = \emptyset \). Since \( Q_\alpha \subset \bigcup \mathcal{M}_\alpha \), we have \( B'' \cap Q_\alpha = \emptyset \). This implies that \( Q_\alpha \) is not abundant everywhere in \( B \), a contradiction!

Having thus established that \( T_\alpha \) is abundant everywhere in \( B \), we can replace \( Q \) with \( Q_\alpha \), \( S \) with \( T_\alpha \), \( T_n \) with \( T_\nu \), \( n_1 \) with \( (\nu_1, \ldots, \nu_k, n_1) \), and \( R_m \) with \( R_{\nu_1 \ldots \nu_k m} \) in the above argument, to obtain the conclusion that \( Q_\alpha \) must be a meager set.

**COROLLARY** (cf. [9]). There is no category base consisting of sets of real numbers for which the sets with the Baire property coincide with the linear Borel sets.

**REMARK.** Concerning more general operations which preserve the classical Baire property and measurability see [1–4, 16, 17].

**REFERENCES**

8. —, *Sur un ensemble non mesurable \( B \)*, J. Math. (9) 2 (1923), 53–72.