PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 100, Number 4, August 1987

CONVEXITY AND BANACH ENVELOPE OF THE WEAK-L_p SPACES

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ABSTRACT. The Banach envelope and a representation of the topological dual of the weak-l_p sequence spaces which involves the Lorentz sequence spaces are computed. The local convexity of weak-L_p spaces is studied also.

w-L_p spaces are function spaces which are closely related to L_p spaces. They were introduced in analysis when it was observed that several important operators such as the Hardy-Littlewood maximal function and the Hilbert transform map L_p into L_p for $p > 1$ but they do not map L_1 into L_1 and rather satisfy the weak condition:

$$\mu\{x\colon (Tf)(x) > y\} \le C\frac{\|f\|}{y}.$$

The space weak-L_p (w-L_p) on a measure space (X, Σ, μ) consists of the measurable function f such that

$$\|f\| = \sup_{\alpha>0}(\alpha^p \mu\{x\colon |f(x)| > \alpha\})^{1/p} < \infty.$$

w-L_p space and its topological dual—avoiding the case when the measure is atomic—have been studied by several authors (see Cwikel, Sagher, and Hunt [**1**, **2**, **5**, **7**]). The Banach envelope and the topological dual of w-L_1 are unknown (although the envelope norm is known), but their properties were studied by Cwikel and Fefferman, and Kupka and Peck (see [**3**, **4**, **9**]).

Our purpose in this paper is to study the Banach envelope and the topological dual of w-L_p when the measure is atomic. Surprisingly, the topological dual turns out to be a classical Lorentz sequence space (see [**10**]); we also show that the Banach envelope is a known space studied by Garling [**6**].

For $0 < p < \infty$ the weak-l_p sequence space is

$$\text{w-}l_p^0 = \left\{x = (x_n)_{n=1}^{\infty}\colon \lim_n n^{1/p} x_n^* = 0\right\}$$

quasi-normed by $\|x\| = \sup_n n^{1/p} x_n^*$, where $(x_n^*)_{n=1}^{\infty}$ is a nonincreasing rearrangement of $(|x_n|)_{n=1}^{\infty}$.

If $v = (v_n)_{n=1}^{\infty}$ is a sequence of real numbers in $c_0 \setminus l_1$ with $1 = v_1 \ge v_2 \ge \cdots \ge v_n \ge \cdots \ge 0$, $d(v, 1)$ denotes the Lorentz space of all sequences $a = (a_n)_{n=1}^{\infty}$ of real numbers such that

$$\|a\|_{v,1} = \sup_{\pi} \sum_{n=1}^{\infty} |a_{\pi(n)}| v_n < \infty,$$

Received by the editors December 3, 1985 and, in revised form, March 5, 1986 and April 30, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46A20, 46A45.

where the supremum is taken in the set of permutations of the integers and it is attained when $(a_{\pi(n)})_{n=1}^{\infty} = (a_n^*)_{n=1}^{\infty}$ (see [**10**]).

If X is a quasi-Banach space whose dual X^* separates the points of X, then X^* is a Banach space under the norm

$$\|x^*\|^* = \sup_{\|x\|\le 1} |x^*(x)|.$$

The closure of X in $(X^{**}, \|\ \|^{**})$ is a Banach space called the Banach envelope of X. X can be identified with its Banach envelope if and only if X is locally convex.

With these preliminaries we can study the w-l_p^0 spaces.

1. LEMMA. *The unit standard vectors $e_n = (\delta_{i,n})_{i=1}^{\infty}$ are a basis of* w-l_p^0.

PROOF. Let $x = (x_n)_{n=1}^{\infty}$ be in w-l_p^0. We can suppose x is nonincreasing and $x_n \ge 0$. We only need observe that

$$\left\| x - \sum_{i=1}^{n} x_i e_i \right\| = \sup_k k^{1/p} x_{n+k} = \sup_k (n+k)^{1/p} x_{n+k},$$

and thus $\lim_n \|x - \sum_{i=1}^n x_i e_i\| = 0$. □

Easy computations show

2. LEMMA. *$l_p \subset$ w-$l_p^0 \subset l_q$ for every $0 < p < q < \infty$, and the inclusion maps are continuous.*

Now we need to point out that every expreme point of the closed unit ball of an n-dimensional w-l_p^0 space is a finite sequence $a = (a_i)_{i=1}^n$ such that

$$(a_i^*)_{i=1}^n = (1, 2^{-1/p}, \dots, n^{-1/p}).$$

Let $y^n = (1, 2^{-1/p}, \dots, n^{-1/p}, 0, 0, \dots)$. We denote by

$$\Pi_n = \{z = (z_k)_{k=1}^{\infty} \colon (z_k^*)_{k=1}^{\infty} = y^n, z_k \ge 0, \text{ and } z_k = 0 \text{ if } k > n\}.$$

3. THEOREM. *The topological dual of* w-l_p^0 *can be identified with the Lorentz sequence space $d(v,1)$ with $v = (v_n)_{n=1}^{\infty}$, $v_n = n^{-1/p}$.*

PROOF. If $0 < p < 1$, then $d(v,1)$ is isomorphic to l_∞ and Lemma 2 ensures that l_∞ is the topological dual of w-l_p^0. If $1 \le p < \infty$, let $f \in (\text{w-}l_p^0)^*$, and put $b = (f(e_i))_{i=1}^{\infty}$, where $(e_i)_{i=1}^{\infty}$ is the unit basis. For every injection π from $\{1, \dots, n\}$ into $\mathbf{N}$, the vector $\sum_{i=1}^n \pm e_{\pi(i)}/i^{1/p}$ has norm one in w-l_p^0 and so

$$\sup_{n,\pi} \sum_{i-1}^{n} i^{-1/p} b_{\pi(i)} \le \|f\|^* \quad \text{and} \quad b \in d(v,1).$$

Conversely, if $b = (b_n)_{n=1}^{\infty}$ belongs to $d(v,1)$, we can define a linear functional f on the linear span of the sequence $(e_i)_{i=1}^{\infty}$ such that $f(e_i) = b_i$ for every i. We denote by $[e_i]_{i=1}^n$ the linear span of $\{e_1, \dots, e_n\}$. On every subspace $[e_i]_{i=1}^n$ f attains its norm at an extreme point of the closed ball, and so we can write

$$\sup_n \sup_\pi \sup_{\substack{a \in [e_{\pi(i)}]_{i=1}^n \\ \|a\| \le 1}} |f(a)| = \sup_n \sup_\pi \sup_{\varepsilon_i = \pm 1} f\left(\sum_{i=1}^n \frac{\varepsilon_i e_{\pi(i)}}{i^{1/p}} \right)$$
$$= \sum_{i=1}^{\infty} i^{-1/p} b_i^* = \|b\|_{v,1}$$

and f admits a continuous linear extension to w-l_p^0 with norm $\le \|b\|_{v,1}$. □

The following result is a description of the Banach envelope of w-l_p^0:

4. PROPOSITION. (a) *If* $0 < p < 1$, l_1 *is the Banach envelope of* w-l_p^0.

(b) *If* $1 \leq p < \infty$, *the Banach envelope of* w-l_p^0 *can be identified with the sequence space* G_p *of all sequences* $x = (x_n)_{n=1}^\infty$ *such that*

$$\lim_n \frac{\sum_{i=1}^n x_i^*}{\sum_{i=1}^n i^{-1/p}} = 0,$$

normed by

$$\|x\| = \sup_n \frac{\sum_{i=1}^n x_i^*}{\sum_{i=1}^n i^{-1/p}},$$

and thus if $1 < p < \infty$, w-l_p^0 *can be identified with* G_p.

PROOF. (a) is direct from Lemma 2.

In order to prove (b) since $(e_n)_{n=1}^\infty$ is a basis of w-l_p^0, its Banach envelope is the closed linear span of $(e_n)_{n=1}^\infty$ in the second dual of w-l_p^0. This closed linear span was computed by Garling [**6**, Theorems 11, 12] concluding the proof of the theorem. □

The galb $G(X)$ of a quasi-Banach space (see [**8**]) is the space of all sequences $(a_n)_{n=1}^\infty$ such that if $x_n \in X$ and $\|x_n\| \leq 1$, then $(\sum_{k=1}^n a_k x_k)$ is bounded. X is said to be galbed by a space of sequences E if $E \subset G(X)$.

A quasi-Banach space is p-convex $(0 < p < 1)$ if it is galbed by l_p. This is equivalent to the existence of a constant A such that

$$\|x_1 + \cdots + x_n\| \leq A(\|x_1\|^p + \cdots + \|x_n\|^p)^{1/p}$$

for $x_1, \ldots, x_n \in X$. And it is said to be log-convex if it is galbed by the Orlicz sequence space l_φ with $\varphi(t) = t(1 + \log^+ 1/t)$. This is equivalent to the existence of a constant A such that

$$\|x_1 + \cdots + x_n\| \leq A\left[\sum_{i=1}^n \|x_i\| \left(1 + \log^+ \frac{\sum_{j=1}^n \|x_j\|}{\|x_i\|}\right)\right]$$

for $x_1, \ldots, x_n \in X$.

The next theorem summarizes the convexity properties of w-L_p. The statement (a) is well known, (b) was partially proved by Kalton in [**8**], and (c) is new, and its proof is inspired by the proof of (b). We will suppose that the functions are defined on $(0, +\infty)$ and μ is the Lebesgue measure:

5. THEOREM. (a) w-L_p *is locally convex if and only if* $p > 1$.

(b) w-L_p *is log-convex if and only if* $p \geq 1$.

(c) *If* $0 < p < 1$, w-L_p *is* q*-convex if and only if* $p \geq q$.

PROOF. (b) Since w-L_1 is log-convex [**8**], we only need to prove that for $0 < p < 1$ w-L_p is not log-convex. Fix n and let

$$\begin{aligned} f_1 &= 1^{-1/p}\chi_{(0,1]} + 2^{-1/p}\chi_{(1,2]} + \cdots + n^{-1/p}\chi_{(n-1,n]}, \\ f_2 &= n^{-1/p}\chi_{(0,1]} + 1^{-1/p}\chi_{(1,2]} + \cdots + (n-1)^{-1/p}\chi_{(n-1,n]}, \\ &\cdots \\ f_n &= 2^{-1/p}\chi_{(0,1]} + 3^{-1/p}\chi_{(1,2]} + \cdots + 1^{-1/p}\chi_{(n-1,n]}. \end{aligned}$$

If $1 \leq i \leq n$, $\|f_i\| = 1$ and $\| \sum_{i=1}^n f_i \| = n^{1/p}(\sum_{i=1}^n i^{-1/p})$. Since

$$\sup_n \frac{\| \sum_{i=1}^n \|f_i\|}{\sum_{i=1}^n \|f_i\| \left(1 + \log^+ \left(\sum_{j=1}^n \|f_j\| / \|f_i\|\right)\right)} = \sup_n \frac{n^{1/p} \sum_{i=1}^n i^{-1/p}}{n(1+\log n)} = \infty,$$

w-L_p is not log-convex.

(c) In order to prove that w-L_p is p-convex, let $f_1, \dots, f_n \in$ w-L_p with $\|f_1\|^p + \cdots + \|f_n\|^p = 1$, and $f = f_1 + \cdots + f_n$. Let $x > 0$ and $A = \{t\colon |f(t)| > x\}$. Let $\mu(A) = \tau$. For $1 \leq i \leq n$ let $E_i = \{t\colon |f_i(t)| > (2/\tau)^{1/p}\}$. Then $\mu(E_i) \leq (\tau/2)\|f_i\|^p$, and thus, if $E = E_1 \cup \cdots \cup E_n$, then $\mu(E) \leq \tau/2$. Following the same steps as Kalton [**8**, Theorem 3.4],

$$\begin{aligned}
\inf_{t \in A} |f(t)| &\leq \frac{2}{\tau} \sum_{i=1}^n \int_{A \setminus E_i} |f_i(t)|\, dt \\
&\leq \frac{2}{\tau} \sum_{i=1}^n \int_A \min\left(|f_i(t)|, \left(\frac{2}{\tau}\right)^{1/p}\right) dt \\
&\qquad\qquad \text{(by [\mathbf{11}, Lemma 3.17, p. 201])} \\
&\leq \frac{2}{\tau} \sum_{i=1}^n \int_0^\tau \min\left(\|f_i\| u^{-1/p}, \left(\frac{2}{\tau}\right)^{1/p}\right) du \\
&= \frac{2}{\tau} \sum_{i=1}^n \left(\int_0^c \left(\frac{2}{\tau}\right)^{1/p} du + \int_c^\tau \|f_i\| u^{-1/p}\, du\right) \quad \left(\text{with } c = \frac{\tau}{2}\|f_i\|^p\right) \\
&= \frac{2}{\tau} \sum_{i=1}^n \left[\left(\frac{2}{\tau}\right)^{-1+1/p} \|f_i\|^p - \frac{\|f_i\|\tau^{1-1/p}}{-1+1/p} + \frac{(\tau/2)^{1-1/p}\|f_i\|^p}{-1+1/p}\right] \\
&\leq \frac{\tau^{-1/p} 2^{1/p}}{1-p}.
\end{aligned}$$

Hence $x(\mu(A))^{1/p} \leq \inf_{t \in A} |f(t)| \tau^{1/p} \leq 2^{1/p}/(1-p)$ and $\|f_1 + \cdots + f_n\| \leq 2^{1/p}/(1-p)$; thus w-$L_p$ is p-convex. The converse can be proved using the technique of (b). □

REMARK. This theorem is also valid when the measure is atomic because a sequence $x = (x_n)_n$ can be regarded as a function f on $(0, \infty)$,

$$f = \sum_{n=1}^\infty x_n \chi_{(n-1,n]},$$

and the norm of x in w-l_p is the same as the norm of f in w-L_p.

We shall remark also that in [**7**, §2] it is proved that w-L_p is r-normed for $r < p$ when $0 < p < 1$ and the measure is not atomic.

ACKNOWLEDGEMENT. The author would like to thank the referee for pointing out some errors in the first version of this paper.

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