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A FORMULA FOR $E_W \exp(-2^{-1}a^2\|x+y\|_2^2)$

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ABSTRACT. We prove that for a complex number a with Re $a^2 > -\pi^2/4$ and $x(\cdot) \in L^2[0,1]$,

$$E_W\{\exp(-2^{-1}a^2\|x+y\|_2^2)\} = (\cosh a)^{-1/2} \exp\left[2^{-1}\left(\int_0^1\int_0^1 k(s,t)x(s)x(t)\,ds\,dt - a^2\int_0^1 x^2(t)\,dt\right)\right],$$

where W, the standard Wiener measure on $C[0,1]$, is the distribution of y and

$$k(s,t) = a^3(2\cosh a)^{-1}[\sinh(a(1-|s-t|)) - \sinh(a(1-|s+t|))].$$

0. Introduction. In this paper we shall prove the following:

THEOREM 1. *Let a be a complex number with* $\operatorname{Re} a^2 > -\pi^2/4$ *and* $x(\cdot) \in L^2[0,1]$. *Then*

(1)
$$E_W\{\exp(-2^{-1}a^2\|x+y\|_2^2)\} = (\cosh a)^{-1/2} \exp\left[2^{-1}\left(\int_0^1\int_0^1 k(s,t)x(s)x(t)\,ds\,dt - a^2\int_0^1 x^2(t)\,dt\right)\right],$$

where W, the standard Wiener measure on $C[0,1]$, is the distribution of y and

(2) $$k(s,t) = a^3(2\cosh a)^{-1}\{\sinh[a(1-|s-t|)] - \sinh[a(1-|s+t|)]\}.$$

This theorem generalizes a well-known formula of Kac [**1**; **3**, p. 101], which treats the case $x(t) \equiv 0$ and a real. We shall state Kac's formula as a corollary.

COROLLARY 2. *For a real,* $E_W\{\exp(-2^{-1}a^2\|y\|_2^2)\} = (\cosh a)^{-1/2}$.

If we let $x(t) \equiv 0$ and $a = ib$, b real, in Theorem 1 and use the fact that $\cosh(ib) = \cos b$, we then have

COROLLARY 3. *Corollary 2 holds for a complex number a with* $\operatorname{Re} a^2 > -\pi^2/4$. *In particular, when $0 \le b \le \pi/2$,* $E_W\{\exp(2^{-1}b^2\|y\|_2^2)\} = (\cos b)^{-1/2}$.

Note that by letting $b \uparrow \pi/2$, $E_W\{\exp(8^{-1}\pi^2\|y\|_2^2)\} = \infty$. This is a well-known result

Theorem 1 is proved in §1. We first treat the case a is real and then extend to the general case by using analytic continuation. Some related formulas are obtained in §2 as simple applications of Theorem 1. We remark that (1) can also be established by an abstract Wiener space set-up.

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1. Proof of Theorem 1. *Case* (i): *a is real.* It is clear from (1) and (2) that we may assume $a \geq 0$.

Let W be the standard Wiener measure on $C[0,1]$ and let $y(t) = \sum_{m=0}^{\infty} y_m g_m(t)$ be the Fourier expansion of $y(\cdot) \in C[0,1]$ with respect to the complete orthonormal system $\{g_m \colon m \geq 0\}$, where $g_m(t) = \sqrt{2}\sin((2m+1)\pi t/2)$. It is well known [**3**] that $y_0, y_1, \ldots$ is a sequence of independent random variables and y_m is normally distributed with mean 0 and variance $\lambda_m = [2/((2m+1)\pi)]^2$.

Write $x(t) = \sum_{m=0}^{\infty} x_m g_m(t)$. Then the left-hand side of (1), which will be denoted by $F(x)$, satisfies

$$\begin{aligned} F(x) &= E_W\left\{\exp\left(-2^{-1}a^2\sum_{m=0}^{\infty}(x_m+y_m)^2\right)\right\} \\ &= \left\{E_W \exp\left[-2^{-1}a^2\sum_{m=0}^{\infty}(y_m^2+2x_my_m)\right]\right\}\exp(-2^{-1}a^2\|x\|_2^2) \\ &= \left\{\prod_{m=0}^{\infty}[E_W\exp(-2^{-1}a^2(y_m^2+2x_my_m))]\right\}\exp(-2^{-1}a^2\|x\|_2^2). \end{aligned}$$

Since y_m is $N(0,\lambda_m)$ distributed, it is easy to compute directly the expectation inside the bracket. This leads to

$$\begin{aligned} F(x) &= \left\{\prod_{m=0}^{\infty}(1+a^2\lambda_m)^{-1/2}\exp(2^{-1}a^4(a^2+\lambda_m^{-1})^{-1}x_m^2)\right\}\exp(-2^{-1}a^2\|x\|_2^2) \\ &= \left[\prod_{m=0}^{\infty}(1+a^2\lambda_m)\right]^{-1/2}\exp\left[2^{-1}\left(\sum_{m=0}^{\infty}a^4(a^2+\lambda_m^{-1})^{-1}x_m^2 - a^2\|x\|_2^2\right)\right]. \end{aligned} \tag{3}$$

It follows from [**3**, p. 101] that the constant can be written as

$$\prod_{m=0}^{\infty}[1+4a^2((2m+1)\pi)^{-2}] = \cosh a. \tag{4}$$

If we define

$$k(s,t) = \sum_{m=0}^{\infty} a^4(a^2+\lambda_m^{-1})^{-1}g_m(s)g_m(t), \tag{5}$$

then

$$\int_0^1\int_0^1 k(s,t)x(s)x(t)\,ds\,dt = \sum_{m=0}^{\infty}a^4(a^2+\lambda_m^{-1})^{-1}x_m^2.$$

This, together with (4), proves the theorem except that we have to check that (2) follows from (5).

Since $g_m(s)g_m(t) = \cos[(2m+1)\pi(s-t)/2] - \cos[(2m+1)\pi(s+t)/2]$ and $\lambda_m = [2/((2m+1)\pi)]^2$, it is clear that (2) is an immediate consequence of the following lemma.

LEMMA 4. *For $t \in [-2,2]$ and a real*

$$\sum_{m=0}^{\infty}\frac{4a^4}{4a^2+(2m+1)^2\pi^2}\cos\frac{(2m+1)\pi t}{2} = a^3(2\cosh a)^{-1}\sinh[a(1-|t|)]. \tag{6}$$

It remains to prove (6). To this purpose we shall state without proof a result [**2**, p. 216] from calculus of residues:

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n \beta s}{\beta^2 s^2 + n^2\pi^2} \cosh \frac{\pi \alpha n}{\beta} = \frac{\cosh(\alpha s)}{\sinh(\beta s)}, \tag{7}$$

where α, β, and s are real numbers such that $|\alpha| < |\beta|$.

Because cos is an even function we need only consider $0 \le t \le 2$ in Lemma 4. If we let $\beta s = 2a$ and $\alpha/\beta = t/2$ in (7) and denote by I_1, I_2 the sum over odd n's and even n's respectively, then

$$I_1 + I_2 = \cosh(at)/\sinh(2a), \tag{8}$$

and by changing n to $2m$ in I_2,

$$\begin{aligned} I_2 &= \sum_{m=-\infty}^{\infty} \frac{2a}{4a^2 + 4m^2\pi^2} \cos(\pi t m) \\ &= \frac{1}{2} \sum_{m=-\infty}^{\infty} \frac{(-1)^m a}{a^2 + m^2\pi^2} \cos(\pi(t-1)m) = \frac{\cosh(a(t-1))}{2\sinh a}, \end{aligned} \tag{9}$$

where the last equation follows from (7) by letting $\beta s = a$ and $\alpha/\beta = t-1$.

As for I_1, it is clear that

$$-I_1 = 2\sum_{m=0}^{\infty} \frac{2a}{4a^2 + (2m+1)^2\pi^2} \cos \frac{(2m+1)\pi t}{2} \tag{10}$$

which shows that $-a^3 I_1$ equals the left-hand side of (6). On the other hand we have from (8) and (9) that

$$\begin{aligned} -I_1 &= (2\sinh a)^{-1}(\cosh(a(t-1))) - (\sinh(2a))^{-1}(\cosh(at)) \\ &= (2\sinh a \cdot \cosh a)^{-1}[(\cosh a)(\cosh(a(t-1))) - \cosh(a + a(t-1))] \\ &= (2\sinh a \cdot \cosh a)^{-1}[-(\sinh a)(\sinh(a(t-1)))] \\ &= (2\cosh a)^{-1}\sinh(a(1-t)) \end{aligned}$$

which, together with (10), proves (6).

Case (ii): *a is a complex number with* $\operatorname{Re} a^2 > -\pi^2/4$. We need only to check that (3), (4), and (6) are still valid in this case. Since

$$|\exp(-2^{-1}a^2\|x+y\|_2^2)| \le \exp(-2^{-1}(\operatorname{Re} a^2)\|x+y\|_2^2)$$

and

$$E_W\{\exp(-2^{-1}(\operatorname{Re} a^2)\|x = y\|_2^2)\} = \lim_n E_W\left\{\exp\left(-2^{-1}(\operatorname{Re} a^2)\sum_{m=0}^{n}(x_m + y_m)^2\right)\right\}$$

by Lebesgue's monotone convergence theorem, we conclude that (3) holds with $\operatorname{Re} a^2 > -\pi^2/4$.

As a pointwise limit of analytic functions the left-hand side of (4) is an analytic function of a. By the uniqueness principle for analytic functions (4) holds for all complex numbers a. Similarly (6) can be checked for a with $\operatorname{Re} a^2 > -\pi^2/4$. This completes the proof.

2. Some applications. Since $\int_0^1 \int_0^1 \sinh[a(1-|s+t|)]\,ds\,dt = 0$,

$$\begin{aligned}\int_0^1 \int_0^1 k(s,t)\,ds\,dt &= a^3(2\cosh a)^{-1} \int_0^1 \int_0^1 \sinh[a(1-|s-t|)]\,ds\,dt \\ &= a^3(2\cosh a)^{-1} \cdot 2 \int_0^1 \int_0^1 \sinh[a(1-t+s)]\,ds\,dt \\ &= a^2 - a(\sinh a)(\cosh a)^{-1}.\end{aligned}$$

The following result follows from Theorem 1 by letting $x(t)$ be a constant function.

COROLLARY 5. *Let x_0 be a real constant and $\operatorname{Re} a^2 > -\pi^2/4$. Then*

$$E_W\left\{\left(-2^{-1}a^2 \int_0^1 (x_0+y(t))^2\,dt\right)\right\} = (\cosh a)^{-1/2} \exp[-x_0^2/(2\sigma^2)], \tag{11}$$

where $\sigma^2 = (a\sinh a)^{-1}\cosh a$.

As a consequence, we have

COROLLARY 6.

$$E_W\left\{\exp\left[-2^{-1}a^2\left(\int_0^1 y^2(t)\,dt - \left(\int_0^1 y(t)\,dt\right)^2\right)\right]\right\} = (a/\sinh a)^{1/2}.$$

PROOF. Since

$$\int_0^1 (x_0+y(t))^2\,dt = \left(x_0 + \int_0^1 y(t)\,dt\right)^2 + \left(\int_0^1 y^2(t)\,dt - \left(\int_0^1 y(t)\,dt\right)^2\right),$$

this corollary can be obtained from a simple computation by integrating both sides of (11) over $x_0 \in (-\infty,\infty)$ and changing the order of integration with E_W.

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