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K-THEORETIC TRIVIALITY FOR RICKART C^*-ALGEBRAS AND $\aleph_0$-CONTINUOUS REGULAR RINGS

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ABSTRACT. In this paper we prove that if R is a purely infinite Rickart C^*-algebra or a purely infinite right $\aleph_0$-continuous regular ring, then R is an infinite sum ring and hence $K_i(R) = 0$ for all i.

1. Introduction. An infinite sum ring is a ring R such that $R \cong 2R$ as right modules and there exists an injective unit preserving ring homomorphism from R to R which satisfies certain properties (see below for definition). Wagoner proved in [**6**] that if R is an infinite sum ring, then $K_i(R) = 0$ for all i.

On the other hand P. Menal and J. Moncasi proved in [**4**] that if R is a purely infinite Rickart C^*-algebra or a purely infinite regular right self-injective ring, then $K_1(R) = 0$. In this paper we extend these results by proving that purely infinite Rickart C^*-algebras and purely infinite right $\aleph_0$-continuous regular rings are infinite sum rings.

2. Notation and background results. All rings considered here are associative with 1. Recall that a ring R is said to be *regular* (von Neumann) if for every $x \in R$ there exists $y \in R$ such that $x = xyx$. In that case, the set of principal right ideals, $L(R_R)$, forms a lattice. If $L(R_R)$ is upper $\aleph_0$-continuous, i.e., every countable subset of $L(R_R)$ has a supremum in $L(R_R)$ and

$$A \wedge \left(\bigvee_{n=1}^{\infty} B_n \right) = \bigvee_{n=1}^{\infty} (A \wedge B_n)$$

for every $A \in L(R_R)$ and every countably ascending chain $B_1 \leqslant B_2 \leqslant \cdots$ in $L(R_R)$, then R is called *right $\aleph_0$-continuous*.

Let R be a ring with involution $*$. An element $e \in R$ is said to be a *projection* if $e^2 = e = e^*$; and R is called a *Rickart $*$-ring* if for all $x \in R$ there exists a projection e generating the right annihilator of x, that is, $\mathbf{r}(x) = eR$. Since R possesses an involution this definition is left-right symmetric so the left annihilator of x, $\mathbf{l}(x)$, is also generated by a projection.

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A *Rickart C^*-algebra* is a C^*-algebra that is also a Rickart*-ring. In that case the set of projections forms an $\aleph_0$-complete lattice, that is every countable set of projections has an infimum and a supremum.

Two idempotents e, f of a ring R are said to be *equivalent* if there exist elements $x \in eRf$, $y \in fRe$ with $xy = e$ and $yx = f$. If e, f are projections in a ring with involution and we can choose $y = x^*$, then e, f are said to be **-equivalent*.

If e, f are idempotents or projections, the notation $e \leqslant f$ means $eR \subseteq fR$ and $e \lesssim f$ means $e \sim g$ for some $g \leqslant f$, where the symbol $\sim$ denotes equivalence for idempotents and *-equivalence for projections. (We refer the reader to [**2**, **3**] for general theory on *-rings and regular rings respectively.)

We state the following lemma which is more or less known for Rickart C^*-algebras.

LEMMA 2.1. *Let R be a right $\aleph_0$-continuous regular ring (Rickart C^*-algebra). Then the following conditions are equivalent*:

(i) *$R \cong 2R$ as right R-modules.*

(ii) *There exists a sequence, $\{e_n\}$, of orthogonal idempotents (projections), all equivalent to* 1 *(*-equivalent) and* $\vee e_n R = R (\vee e_n = 1)$.

PROOF. Suppose R is a right $\aleph_0$-continuous regular ring and assume that $R \cong 2R$ as right R-modules. Then we can construct a sequence, $\{f_n\}$, of orthogonal idempotents, all equivalent to 1. Set $fR = \vee f_n R$. Now $fR \leqslant R$ and $R \lesssim fR$, hence by the general Schröder-Bernstein theorem [**1**, Lemma 2.10], $R \cong fR$, so there exists a ring isomorphism $\varphi\colon fRf \to R$. Taking $e_n = \varphi(f_n)$ we have the desired sequence.

Conversely we have $2R \lesssim \vee e_n R = R$ and again by [**1**, Lemma 2.10] $R \cong 2R$.

The proof for Rickart C^*-algebras goes similarly since the general Schröder-Bernstein theorem holds [**2**, Proposition 2.12.1]. □

A Rickart C^*-algebra or a right $\aleph_0$-continuous regular ring is said to be *purely infinite* if it satisfies one of the equivalent conditions of Lemma 2.1.

If R is any ring satisfying $R \cong 2R$ as right R-modules, then there exist elements $\alpha_i, \beta_j \in R$, $i, j = 0, 1$, such that the matrices

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}, \qquad (\beta_0 \quad \beta_1)$$

are mutually inverse, i.e.,

$$\alpha_0\beta_0 = \alpha_1\beta_1 = 1, \qquad \beta_0\alpha_0 + \beta_1\alpha_1 = 1.$$

If for a particular choice of these elements there exists a unit preserving ring homomorphism $\infty\colon R \to R$ such that

$$\beta_0 r \alpha_0 + \beta_1 r^\infty \alpha_1 = r^\infty$$

for every $r \in R$, then R is called an *infinite sum ring* (see [**6**, p. 355]).

3. The results. If R is a purely infinite Rickart C^*-algebra, then by Lemma 2.1, R is ring isomorphic to $M_2(R)$, so R is semihereditary and we have the following result noted by Handelman that will be used in Lemma 3.2.

LEMMA 3.1 (HANDELMAN). *A semihereditary Rickart C^*-algebra has polar decomposition.*

PROOF. See, for example, [5, Lemma 3.5]. □

LEMMA 3.2. *Let R be a purely infinite Rickart C^*-algebra. Then for every sequence, $\{e_n\}_{n\geqslant 0}$, of orthogonal projections all *-equivalent to 1 with $\bigvee e_n = 1$, there exist elements $\alpha_i, \beta_j \in R$, $i, j = 0, 1$, such that*

$$e_n = \beta_1^n \beta_0 \alpha_0 \alpha_1^n$$

and the matrices

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}, \qquad (\beta_0 \quad \beta_1)$$

are mutually inverse.

PROOF. Fix $x_n \in e_n R$ such that

$$x_n x_n^* = e_n, \qquad x_n^* x_n = 1 \quad \text{for all } n \geqslant 0.$$

Set $F = \bigoplus x_n R$. Thus F is a countable free right ideal of R with basis $\{x_n\}$. Let S be the idealizer of F in R. Every element $x \in S$ defines by left multiplication an endomorphism $\hat{x}$ of F. Clearly $x \to \hat{x}$ defines an injective ring homomorphism $\hat{}$: $S \to \mathrm{End}_R(F)$. Let $\varphi, \psi \in \mathrm{End}_R(F)$ be the homomorphisms given by the rules

$$\varphi(x_n) = x_{n+1} \quad \text{for all } n \geqslant 0,$$
$$\psi(x_n) = x_{n-1} \quad \text{for all } n \geqslant 1 \text{ and } \psi(x_0) = 0.$$

We claim that φ, ψ are given by left multiplication.

Let $a, b \in R$ be the elements defined respectively by the norm convergent series

$$\sum_{n\geqslant 0} (e_n/2^n), \qquad \sum_{n\geqslant 0} (x_{n+1}x_n^*/2^n).$$

Then $a \geqslant 0$ and $a^2 = b^*b$, so $a = (b^*b)^{1/2}$. By Lemma 3.1, R has polar decomposition, so we obtain from [2, Proposition 4.21.3] a partial isometry w satisfying

$$RP(b) = w^*w, \qquad LP(b) = ww^*$$

with $b = wa$, $a = w^*b$. Now we have for all $n \geqslant 0$,

$$(x_{n+1}/2^n) = bx_n = wax_n = w(x_n/2^n),$$
$$(x_n/2^n) = ax_n = w^*bx_n = w^*(x_{n+1}/2^n).$$

On the other hand $x_0^*b = 0$ so $x_0^*ww^* = 0$ and since the involution is proper, right multiplication by x_0 yields $w^*x_0 = 0$. Hence φ, ψ are given by left multiplication by w, w^* respectively as claimed.

Now

$$w^n x_0 x_0^* (w^*)^n = x_n x_n^* = e_n$$

and observe that left multiplication by w^*w and $e_0 + ww^*$ gives the identity over F. So defining

$$\alpha_0 = x_0^*, \quad \beta_0 = x_0, \quad \alpha_1 = w^*, \quad \beta_1 = w$$

the matrices

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}, \qquad (\beta_0 \quad \beta_1)$$

are mutually inverse which completes the proof. □

The following lemma is the analogue of Lemma 3.2 for right $\aleph_0$-continuous regular rings.

LEMMA 3.3. *Let R be a purely infinite right $\aleph_0$-continuous regular ring. Then there exist elements $\alpha_i, \beta_j \in R$, $i, j = 0, 1$, such that the matrices*

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}, \qquad (\beta_0 \quad \beta_1)$$

are mutually inverse and $e_n = \beta_1^n \beta_0 \alpha_0 \alpha_1^n$ for $n \geqslant 0$ is a sequence of orthogonal idempotents all equivalent to 1 and $\bigvee e_n R = R$.

PROOF. Let $\{f_n\}_{n \geqslant 0}$ be a sequence of orthogonal idempotents, all equivalent to 1 with $\bigvee f_n R = R$ which exists by Lemma 2.1.

Fix $x_n, y_n \in R$ such that

$$x_n y_n = f_n, \quad y_n x_n = 1 \quad \text{for all } n \geqslant 0.$$

Set F, S, φ, ψ as in Lemma 3.2. Now since R is purely infinite we see by [**3**, Corollary 14.13] that R is right $\aleph_0$-injective, so $S \cong \mathrm{End}_R(F)$ and hence φ, ψ are given by left multiplication by some elements β_1, α_1 respectively.

Now if we define $\alpha_0 = y_0$ and $\beta_0 = x_0$ it is easily seen that the matrices

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}, \qquad (\beta_0 \quad \beta_1)$$

are mutually inverse, and observing that

$$\beta_1^n \beta_0 \alpha_0 \alpha_1^n R = \beta_1^n \beta_0 R = \beta_1^n x_0 R = x_n R = f_n R$$

the proof is complete. □

THEOREM. *Let R be a purely infinite Rickart C^*-algebra or a purely infinite right $\aleph_0$-continuous regular ring. Then R is an infinite sum ring.*

PROOF. Suppose R is a purely infinite Rickart C^*-algebra. Let $\{e_n\}_{n \geqslant 0}$ be a sequence of projections as in Lemma 3.2 and put $x_n = \beta_1^n \beta_0$. Then $x_n^* = \alpha_0 \alpha_1^n$, $e_n = x_n x_n^*$, and $x_n^* x_n = 1$ for all $n \geqslant 0$. Set F, S as in the lemma and consider the ring homomorphism

$$\sim\ : R \to \mathrm{End}_R(F)$$
$$r \to \tilde{r}\colon x_n \to x_n r.$$

Now $\tilde{r}(e_n) = e_n \tilde{r}(e_n) e_n$. By [**2**, Proposition 1.10.3] applied to the centralizer of $\{e_n\}$, we see that $\tilde{r}$ is given by left multiplication. Hence we can define a unit preserving ring homomorphism $\infty\colon R \to R$ by the rule

$$r^\infty = \Lambda^{-1}(\tilde{r}).$$

Now

$$(\beta_0 r\alpha_0 + \beta_1 r^\infty \alpha_1)x_n = \begin{cases} \beta_0 r = x_0 r & \text{if } n = 0, \\ \beta_1 r^\infty x_{n-1} = \beta_1 x_{n-1} r = x_n r & \text{if } n > 0, \end{cases}$$

so $\alpha_0 r\beta_0 + \beta_1 r^\infty \alpha_1 = r^\infty$ which completes the proof for Rickart C^*-algebras.

For purely infinite right $\aleph_0$-continuous regular rings, the proof is similar. In this case, one uses Lemma 3.3 instead of Lemma 3.2, and since $S \cong \mathrm{End}_R(F)$, the result follows. □

COROLLARY. *Let R be a purely infinite Rickart C^*-algebra or a purely infinite right $\aleph_0$-continuous regular ring. Then $K_i(R) = 0$ for all $i \in \mathbf{Z}$.*

PROOF. [**6**, Corollary 2.5]. □

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