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THE GENERALIZED BURGERS' EQUATION AND THE NAVIER-STOKES EQUATION IN $\mathbf{R}^n$ WITH SINGULAR INITIAL DATA

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ABSTRACT. From an abstract theory of Weissler we construct a simple local existence theory for a generalization of Burgers' equation and the Navier-Stokes equation in the Banach space $L^p(\mathbf{R}^n)$. Our conditions on p recover the conditions of Giga and Weissler in the latter case except for the borderline situation $p = n$. For the generalized Burgers' equation our results are apparently new; moreover we show that these local solutions are in fact global solutions in this case. We also obtain results for the generalized Burgers' equation with $\mathbf{R}^n$ replaced by a bounded domain Ω with smooth boundary. Using a somewhat more complex abstract theory of Weissler, we are able to improve on our results found in the case $\Omega = \mathbf{R}^n$, and also obtain global existence.

1. Introduction. Semilinear parabolic equations over domains Ω in $\mathbf{R}^n$ with singular initial data have been studied recently by Giga, Weissler, and others (see e.g., [**5–7**, **9–12**] for a partial list). In [**10**] for example, the existence of unique local mild solutions of the equation

$$u_t - \Delta u = |u|^\alpha u \tag{1.1}$$

for $\alpha > 0$ was established for any initial data in $L^p(\Omega)$ where p depended on α and n and Ω was a bounded domain with smooth boundary. The subsequent work of Giga and Weissler cited above produced similar results for (1.1) when $\Omega = \mathbf{R}^n$ and also treated the Navier-Stokes equation for incompressible flow when Ω is a half-plane [**11**], in bounded domains with smooth boundary [**5**, **6**], and, quite recently, in $\mathbf{R}^n$ [**5**].

In the case of the Navier-Stokes equation, these results establish, in particular, local existence and uniqueness of solutions for any initial data in $E^p(\Omega)$, the solenoidal subspace of $L^p(\Omega)$, provided $p \geqslant n$.

In this paper we demonstrate how simply and effectively the original abstract theory of [**10**] applies to a generalized Burgers' equation, and the Navier-Stokes equation, when $\Omega = \mathbf{R}^n$. In the latter case we obtain results similar to those cited above in [**5**], except for the borderline case $p = n$. Our results for the generalized Burgers' equation are apparently new.

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The original Burgers' equation is

$$u_t - u_{xx} + uu_x = 0, \tag{1.2}$$

where $u = u(x,t)$ and $x \in \mathbf{R}^1$. The generalization considered here is

$$u_t - \Delta u + \operatorname{div}\psi(u) = 0, \tag{1.3}$$

where $u = u(x,t)$ with $x \in \mathbf{R}^n$, $\Delta = \sum_{j=1}^n \partial^2/(\partial x_j)^2$, and $\psi = (\psi_1, \ldots, \psi_n)$ is a function from $\mathbf{R}$ to $\mathbf{R}^n$ such that the components ψ_i are polynomials of degree γ or less. In §2 we use the abstract theory of [**10**] to establish local existence and uniqueness of mild solutions of (1.3) for any initial data $u(x,0) \equiv u_0$ in $L^p(\Omega)$ whenever

$$p > (\gamma - 1)n. \tag{1.4}$$

Note that (1.4) reduces to $p > n$ when $\gamma = 2$. We show that these solutions are *global* solutions in §3, using results found in [**2**].

In the references cited above, the Navier-Stokes equation for incompressible flow is written in the form

$$u_t - P\Delta u = -P(u, \nabla)u, \tag{1.5}$$

where for fixed $t \geqslant 0$, $u = (u^1, \ldots, u^n)$ is a map from Ω to $\mathbf{R}^n$, P is the projection onto solenoidal vectors, and

$$(u, \nabla) = \sum_{j=1}^n u^j(\partial/\partial x_j). \tag{1.6}$$

In §4 we discuss (1.5) for $\Omega = \mathbf{R}^n$, and establish local existence and uniqueness of mild solutions for arbitrary initial data in $E^p(\mathbf{R}^n) \equiv PL^p(\mathbf{R}^n)$ whenever $p > n$. The fact that P commutes with Δ when $\Omega = \mathbf{R}^n$ will play a crucial role in our analysis.

In §5 we consider the extension of the results of §2 to bounded domains. The author has recently been informed that [**4**], while the results of §2 in this paper are apparently new, the abstract theory of [**5**] can be applied when Ω is bounded with smooth boundary to obtain local existence for $p \geqslant (\gamma - 1)n$, thus adding the borderline case to (1.4). We show in §5 that another abstract theorem of Weissler found in [**11**] is applicable to the bounded domain case, and also yields the result $p \geqslant (\gamma - 1)n$.

We conclude this section by outlining the abstract setting of Weissler that will be applied in §§2 and 4. Let E be a Banach space with norm $\|\cdot\|$ and let e^{tA} be a C_0 semigroup on E. For each $t > 0$ let K_t: $E \to E$ be a semi-Lipschitz map, i.e., if U_α is the closed ball of radius $\alpha > 0$ in E, then K_t is Lipschitz continuous over each U_α with Lipschitz constant $C_\alpha(t)$. The following is an equivalent form of a portion of Theorem 1 of [**10**].

THEOREM 1.1. *Let E, e^{tA}, K_t, and $C_\alpha(t)$ be as above and assume further that*:

(a) $e^{sA}K_t = K_{s+t}$ *for* $s, t > 0$.

(b) *For each* $\alpha > 0$, $C_\alpha(\cdot)$ *is in* $L^1(0, \varepsilon)$ *for some* $\varepsilon > 0$.

(c) $t \mapsto \|K_t(0)\|$ *is in* $L^1(0, \varepsilon)$ *for some* $\varepsilon > 0$.

Then for each $\phi \in E$ *there exists a* $T_\phi > 0$ *such that the integral equation*

$$u(t) = e^{tA}\phi + \int_0^t K_{t-s}(u(s))\, ds \tag{1.7}$$

has a unique local solution $u \in C([0, T_\phi); E)$.

In Weissler's original application $K_t(\phi) = e^{tA}J(\phi)$, corresponding to the equation

$$u_t = Au + J(u), \tag{1.8}$$

where $E = L^q(\Omega)$ for an appropriate q, A is a type of elliptic differential operator and J is a polynomial. The choice of q depends on n and the degree of J; this dependence arises from properties of A and the Sobolev embedding theorems (see, e.g., [**1**]). Our choice of p in §§2 and 4 will come from similar considerations, once we select appropriate choices for K_t in each case.

2. Local existence for the generalized Burgers' equation. We seek local mild solutions of (1.3). The corresponding integral equation is

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\operatorname{div}(\psi(u(s)))\,ds. \tag{2.1}$$

Noting that $\partial/\partial x_j$ commutes with Δ over $\mathbf{R}^n$, $j = 1, \ldots, n$, we rewrite (2.1) in the form

$$u(t) = e^{t\Delta}u_0 + \int_0^t \operatorname{div}\left(e^{(t-s)\Delta}\psi(u(s))\right)ds. \tag{2.2}$$

To apply Theorem 1.1 to (2.2), we set

$$K_t(\phi) = \operatorname{div}\left(e^{t\Delta}\psi(\phi)\right), \tag{2.3}$$

where $\phi \in L^p(\mathbf{R}^n)$; the following theorem establishes the choice of p.

THEOREM 2.1. *For $\gamma \geqslant 2$ and $p > \max\{\gamma, (\gamma - 1)n\}$ the hypotheses of Theorem* 1.1 *are satisfied with K_t as in* (2.3), *$E = L^p(\mathbf{R}^n)$, and $e^{tA} = e^{t\Delta}$.*

In other words, (2.2) has a unique local solution $u \in C([0, T);\ L^p(\mathbf{R}^n))$ for some $T > 0$. Recall that γ is the maximum degree of the polynomials forming the components of ψ.

PROOF OF THEOREM 2.1. We first consider the case where each ψ_i is a monomial of degree γ. For $m > 0$ and $1 \leqslant p < +\infty$ let $W^{m,p}(\mathbf{R}^n)$ denote the set of measurable functions whose derivatives up to order m are in $L^p(\mathbf{R}^n)$. A suitable norm that makes $W^{m,p}(\mathbf{R}^n)$ a Banach space is $\|f\|_{m,p} \equiv \|f\|_p + \|(-\Delta)^{m/2}f\|_p$. Note that m can be any positive real number (see Chapter VII of [**1**]). If $F = (f_1, \ldots, f_m)$ is vector-valued we let $\|F\|_{m,p}$ denote $\sup_i \|f_i\|_{m,p}$.

By Hölder's inequaltiy every monomial of degree γ gives rise to a semi-Lipschitz map of $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ where $p > \gamma$ and $q = p/\gamma$. Hence for each $\alpha > 0$ let U_α denote the closed ball of radius α in $L^p(\mathbf{R}^n)$ and let M_α be a constant so that

$$\|\psi(v) - \psi(w)\|_{p/\gamma} \leqslant M_\alpha \|v - w\|_p \tag{2.3}$$

for all $v, w \in U_\alpha$.

Let K_1 be a constant such that $\|\operatorname{div}(F)\|_p \leqslant K_1\|F\|_{1,p}$ whenever $F = (f_1, \ldots, f_n)$ with $f_i \in W^{1,p}(\mathbf{R}^n)$. By the Sobolev embedding theorems, for each $m > 0$ there exists a constant C depending only on n, p, and m such that $\|f\|_{1,p} \leqslant C\|f\|_{(m+1),p/\gamma}$ provided $p = n(\gamma - 1)/m$; hence for $F = (f_1, \ldots, f_n)$ with $f_i \in L^{p/\gamma}(\mathbf{R}^n)$ we have

(for the above choice of p) that

$$\|\operatorname{div}(e^{t\Delta}F)\|_p \leqslant K_1\|e^{t\Delta}F\|_{1,p} \leqslant K_1C\|e^{t\Delta}F\|_{(m+1),p/\gamma} = K_1C\left[\|e^{t\Delta}F\|_{p/\gamma} + \|(-\Delta)^r e^{t\Delta}F\|_{p/\gamma}\right], \tag{2.4}$$

where $r = (m+1)/2$. Now $e^{t\Delta}$ is an analytic contraction semigroup on $L^{p/\gamma}(\mathbf{R}^n)$, hence from (2.4) we have

$$\|\operatorname{div}(e^{t\Delta}F)\|_p \leqslant K_1C(1 + dt^{-r})\|F\|_{p/\gamma} \tag{2.5}$$

for some constant d, independent of F.

Thus if $v, w \in U_\alpha$ we have by (2.3) and (2.5) that

$$\begin{aligned}\|K_t(v) - K_t(w)\|_p &= \|\operatorname{div}(e^{t\Delta}(\psi(v) - \psi(w)))\|_p \\ &\leqslant K_1C(1 + dt^{-r})\|\psi(v) - \psi(w)\|_{p/\gamma} \\ &\leqslant M_\alpha K_1C(1 + dt^{-r})\|v - w\|_p\end{aligned} \tag{2.6}$$

for each $m > 0$, provided $p = n(\gamma - 1)/m$. Note that $r < 1$ if $m < 1$, which will occur provided $p > n(\gamma - 1)$. With this choice of p part (b) of Theorem 1.1 is verified with $C_\alpha(t) = M_\alpha K_1C(1 + dt^{-r})$. For the case when each ψ_i is a polynomial of degree γ or less, we can write $K_t(\phi)$ as a sum of terms of the form $\partial/(\partial x_j)e^{t\Delta}(\bar{\psi}(\phi))$, where $\bar{\psi}$ is a monomial; we now proceed as above, but choose C and m precisely for each term. The critical choice of m will be for the terms of degree γ; for other terms we can choose m (and hence r) to be smaller, so that $C_\alpha(t)$ will now be a sum of terms integrable near zero. This establishes part (b) for the general case.

Meanwhile part (c) of Theorem 1.1 is trivial since the components of $\psi(0)$ are constants, while (a) follows since $e^{t\Delta}$ commutes with $\partial/\partial x_j$, $j = 1, \ldots, n$. This completes the proof of Theorem 2.1.

It is evident that if $-\Delta$ is replaced by $A \equiv (-\Delta)^m$ for $m > 1$, then we might be able to allow for a lower p. The following is a corollary of the proof of Theorem 2.1.

THEOREM 2.2. *Let A be as in the preceding paragraph. Then for $K_t(\cdot) = \operatorname{div}(e^{-tA}\psi(\cdot))$ and $E = L^p(\mathbf{R}^n)$ the conditions of Theorem* 1.1 *are satisfied provided* $p > \max\{\gamma, n(\gamma - 1)/(2m - 1)\}$.

3. Global existence for the generalized Burgers' equation. If (1.3) has initial data in $W^{1,p}(\mathbf{R}^n)$ with $p > n$, then it was shown in **[2]** that (1.3) has a unique global solution $u \in C([0, +\infty); W^{1,p}(\mathbf{R}^n))$. Although the analysis in **[2]** is carried out for bounded domains (and a generalized version of (1.3)) we noted at that time that the estimates used, primarily the imbedding of $W^{1,p}(\mathbf{R}^n)$ into $C_B(\mathbf{R}^n)$ for $p > n$ and the maximum principle for linear equations of the form

$$w_t = \Delta w + a(x) \cdot \nabla w \tag{3.1}$$

(where the components of a are in $L^\infty(\mathbf{R}^n)$), carry over to the case where $\Omega = \mathbf{R}^n$. In particular, by Proposition (3.1) of **[2]** we have the estimate

$$\|u(t)\|_\infty \leqslant \|u_0\|_\infty, \qquad t \in [0, +\infty), \tag{3.2}$$

if $u_0 \equiv u(0) \in W^{1,p}(\mathbf{R}^n)$.

If $u_0 \in L^p(\mathbf{R}^n)$ with $p > (\gamma - 1)n$, we established the existence of a unique local solution $u \in C([0,T);\ L^p(\mathbf{R}^n))$ for some $T > 0$ in §2. By conclusion (v) of Theorem 1 of [**10**], to establish global existence it suffices to show that $\|u(t)\|_p$ does not blow up in finite time. For this purpose assume first that $u_0 \in L^p(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$. Let $u_0^n = \exp(n^{-1}\Delta)u_0$; then by the maximum principle and the contractive properties of $e^{t\Delta}$,

$$\|u_0^n\|_\infty \leqslant \|u_0\|_\infty, \qquad \|u_0^n\|_p \leqslant \|u_0\|_p \tag{3.3}$$

while $u_0^n \to u_0$ in $L^p(\mathbf{R}^n)$ since $e^{t\Delta}$ is a strongly continuous semigroup. Moreover, $u_0^n \in W^{1,p}(\mathbf{R}^n)$ since $e^{t\Delta}$ is analytic on $L^p(\mathbf{R}^n)$.

Let u_n denote the solution of (1.3) with initial data u_0^n. Note $\operatorname{div}\psi(u_n) = g(u_n) \cdot \nabla u_n$, where $g(r) = \psi'(r)$, $r \in \mathbf{R}$. From (3.2) and (3.3) there exists a constant M, independent of t and n, such that

$$\|g(u_n)\|_\infty \leqslant M. \tag{3.4}$$

Rewriting (2.2) for u_n in the form

$$u_n(t) = e^{t\Delta}u_0^n + \int_0^t e^{(t-s)\Delta}g(u_n(s)) \cdot \nabla u_n(s)\, ds, \tag{3.5}$$

we apply ∇ to both sides and take L^p-norms to obtain

$$\|\nabla u_n(t)\|_p \leqslant dt^{-1/2}\|u_0^n\|_p + \int_0^t d(t-s)^{-1/2}M\|\nabla u_n(s)\|_p\, ds, \tag{3.6}$$

where d is as in §2 and M is as in (3.4). Multiplying both sides by $t^{1/2}$ we obtain

$$t^{1/2}\|\nabla u_n(t)\|_p \leqslant d\|u_0^n\|_p + t^{1/2}\int_0^t C(t,s)\left[s^{1/2}\|\nabla u_n(s)\|_p\right] ds, \tag{3.7}$$

where $C(t,s) = dM(t-s)^{-1/2}s^{-1/2}$. Applying Gronwall's inequality and (3.3) to (3.7) we see that

$$t^{1/2}\|\nabla u_n(t)\|_p \leqslant d\|u_0\|_p \exp\left(T^{1/2}D(t)\right) \tag{3.8}$$

for $t \in [0,T)$, where $D(t) = \int_0^t C(t,s)\, ds$.

Hence, with $E(t) \equiv \exp(T^{1/2}D(t)) \in C([0,T])$,

$$\begin{aligned} \|u_n(t)\|_p &\leqslant \|u_0^n\|_p + \int_0^t Md\|u_0\|_p s^{-1/2}E(s)\, ds \\ &\leqslant \|u_0\|_p + 2t^{1/2}Md\|u_0\|_p \sup_{0 \leqslant t \leqslant T} E(t). \end{aligned} \tag{3.9}$$

Given a closed interval of existence $[0,T_1]$ for u, by (3.9) we can choose α independent of n so that $\|u^n(t)\|_p \leqslant \alpha$ and $\|u(t)\|_p \leqslant \alpha$ for $t \in [0,T_1]$. Then from §2 we have, for $0 \leqslant t \leqslant T_1$,

(3.10)

$$\begin{aligned} \|u_n(t) - u(t)\|_p &\leqslant \|e^{t\Delta}(u_0^n - u_0)\|_p + \int_0^t \left\|\operatorname{div}\left(e^{t\Delta}(\psi(u_n(s)) - \psi(u(s)))\right)\right\|_p ds \\ &\leqslant \|u_0^n - u_0\|_p + \int_0^t C_\alpha(s)\|u_n(s) - u(s)\|_p\, ds, \end{aligned}$$

so by Gronwall's inequality

$$\|u_n(t) - u(t)\|_p \leqslant \|u_0^n - u_0\|_p \exp(D_\alpha(t)), \tag{3.11}$$

where $D_\alpha(t) = \int_0^t C_\alpha(s)\,ds$. Thus $\lim_{n\to\infty} u_n(t) = u(t)$ uniformly in $L^p(\mathbf{R}^n)$ on each such $[0, T_1]$. In particular (3.9) must be satisfied with $u(t)$ in place of $u_n(t)$ on each such $[0, T_1]$, hence $\|u(t)\|_p$ cannot blow up in finite time. We have thus proven the following result.

PROPOSITION 3.1. *The solutions found in* §2 *are global solutions if* $u_0 \in L^p(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$.

We now use a bootstrap argument that employs Proposition 3.1. Let $r \in (0,1)$ and $p > (\gamma - 1)n$ be numbers to be determined with $n \geqslant 2$.

Let $T > 0$ be such that a local solution $u(t)$ of (1.3) exists for $t \in [0, T)$ as in Theorem 2.1 (note T depends on p). Then for $t \in (0, T)$ and K_1 as in §2,

$$\begin{aligned}(3.11)\quad &\|(-\Delta)^{r/2}u(t)\|_p \\ &\leqslant dt^{-r/2}\|u_0\|_p + K_1\Big[\int_0^t \|(-\Delta)^{r/2}e^{(t-s)\Delta}u(s)\|_p \\ &\qquad + \|(-\Delta)^{r/2}(-\Delta)^{1/2}e^{(t-s)\Delta}\psi(u(s))\|_p\,ds\Big] \\ &\leqslant dt^{-r/2}\|u_0\|_p + K_1C\Big[\int_0^t d(t-s)^{-r/2}\|u(s)\|_{p/\gamma}\,ds \\ &\qquad + \int_0^t \|(-\Delta)^{(\alpha+r+1)/2}e^{(t-s)\Delta}\psi(u(s))\|_{p/\gamma}\,ds\Big],\end{aligned}$$

where $\alpha = (\gamma - 1)n/p$ and C depends on n, p, and α. The right-hand side of (3.11) will be finite provided $\alpha + r < 1$, or

$$(3.12)\qquad p > (\gamma - 1)n/(1 - r) > (\gamma - 1)n.$$

Choose p such that (3.12) is satisfied and fix $t = t_0$ in $(0, T)$. Then $u(t_0) \in W^{r,p}(\mathbf{R}^n)$. Now $W^{r,p}(\mathbf{R}^n) \subset C_B(\mathbf{R}^n)$ provided

$$(3.13)\qquad p > n/r.$$

As $(\gamma - 1)n/(1 - r)$ and n/r are increasing and decreasing functions, respectively, on $(0,1)$ the minimum condition on p given by (3.12) and (3.13) occurs when $(\gamma - 1)n/(1 - r) = n/r$, which happens when $r = 1/\gamma$. For such an r, (3.12) and (3.13) become

$$(3.14)\qquad p > \gamma n.$$

Then with p and r as above, let $v(t)$ solve (1.3) with $u_0 = u(t_0)$; by Theorem 3.1 v is a continuous $L^p(\mathbf{R}^n)$-valued *global* solution. Let

$$(3.15)\qquad w(t) = \begin{cases} u(t), & 0 \leqslant t \leqslant t_0, \\ v(t - t_0), & t \geqslant t_0. \end{cases}$$

Then $w \in C([0, +\infty);\ L^p(\mathbf{R}^n))$ is a global solution of (1.5), which by uniqueness equals $u(t)$ on all of $[0, T)$. Thus $\|u(t)\|_p$ cannot blow up in a finite time, and so u is a global solution (which equals w on $[0, +\infty)$).

Let $u = u_1$, $p = p_1$, and $r = r_1$. Then we define our bootstrap procedure inductively as follows: Suppose we have a global solution $u_k \in C([0, +\infty); L^p(\mathbf{R}^n))$ of (1.3) with $p = p_k$. Then for $r_{k+1} \in (0,1)$ and $p_{k+1} \in ((\gamma - 1)n, p_k)$ to be determined we have that (3.11) holds if (3.12) is satisfied with p replaced by p_{k+1}, r by r_{k+1}, and u by u_{k+1} (where we let $u_{k+1} \in C([0, T_{k+1}); L^q\mathbf{R}^n)$ be the local solution of (1.3) with $u_0 \in L^q(\mathbf{R}^n)$, $q = p_{k+1}$). Then for a fixed $t_0 \in (0, T_{k+1})$ we have that $u_{k+1}(t_0) \in L^p(\mathbf{R}^n)$, $p = p_k$, provided $p_{k+1} > (\gamma^{k+1}n)/(\gamma^{k+1}r_{k+1} + \gamma^k + \cdots + 1)$, since $p_k = (\gamma^k n)/(\gamma^{k-1} + \cdots + 1)$.

If

$$r_{k+1} = 1/\gamma^{k+1} \tag{3.16}$$

then the two conditions on p_{k+1} now require that

$$p_{k+1} > (\gamma^{k+1}n)/(\gamma^k + \gamma^{k-1} + \cdots + 1). \tag{3.17}$$

The argument with v and w above applies, with v satisfying (1.3) with u_0 replaced by $u_{k+1}(t_0)$, and $u(t)$ replaced by $u_{k+1}(t)$, $t \in (0, T_{k+1})$. We thus obtain global solutions $u_{k+1} \in C([0, +\infty); L^q(\mathbf{R}^n))$ with $q = p_{k+1}$ for all positive integers k provided p_{k+1} satisfies (3.17).

Writing γ^{k+1} as $(\gamma - 1) + 1$ we then see that the right-hand side of (3.17) converges to $(\gamma - 1)n$. Thus for any $p > (\gamma - 1)n$, we have $p \geqslant p_{k+1} > (\gamma - 1)n$ for large enough k; this establishes the following theorem.

THEOREM 3.1. *The solutions found in* §2 *are global, i.e., for* $u_0 \in L^p(\mathbf{R}^n)$ *with* $p > (\gamma - 1)n$, (1.3) *has a unique solution* $u \in C([0, +\infty); L^p(\mathbf{R}^n))$.

We have proven Theorem 3.1 with $n \geqslant 2$. The details with $n = 1$ are similar; note in that case $\max\{\gamma, (\gamma - 1)n\} = \gamma$. Also note that by the proof of Theorem 3.1 $u(t) \in C_B(\mathbf{R}^n)$ for any $t \in (0, +\infty)$. In fact, a simple integral equation argument shows that $u \in C((0, +\infty); C_B(\mathbf{R}^n))$.

4. The Navier-Stokes equation in $\mathbf{R}^n$. As has frequently been done before (see e.g., [**3**, **5**, **11**]) we write the Navier-Stokes equation in the form

$$u_t - Au = -P(u, \nabla)u, \tag{4.1}$$

where P is the projection onto the solenoidal vectors, $A = P\Delta$, and the ith component of $P(u, \nabla)u$ is

$$P\left(\sum_{j=1}^{n} u_j(\nabla_j u_j)\right), \tag{4.2}$$

where $u = (u_1, \ldots, u_n)$ and $\nabla_j = \partial/(\partial x_j)$. Let E_p be the closure in $(L^p(\mathbf{R}^n))^n$ of $\{u \in (C_0(\mathbf{R}^n))^n \mid \operatorname{div} u = 0\}$. Then it is well known [**3**] that P is a continuous map from $(L^p(\mathbf{R}^n))^n$ to E_p when $p = 2$. The same is true, however, for $p \geqslant 1$ (see [**6**, p. 890 or **5**]). Meanwhile, since we are considering (4.1) over $\mathbf{R}^n$, we can take $A = \Delta$ since P commutes with Δ in this case.

For our purposes here we rewrite (4.2) in the form

$$P\left(\sum_{j=1}^{n} \nabla_j(u_j u_i)\right). \tag{4.3}$$

This is equivalent to (4.2) since a solution u of (4.1) must satisfy $\operatorname{div} u = 0$.

The corresponding integral solution for (4.1) is

$$u(t) = e^{tA}u_0 + \int_0^t K_{t-s}(u(s))\,ds, \tag{4.4}$$

where $K_t(u) = -e^{tA}P(u,\nabla)u$. We want to apply Theorem 1.1 to (4.4) with this choice of K_t, $E = E_p$, and $e^{tA} = e^{t\Delta}$. Since P and ∇_j both commute with $A = \Delta$ in E, we can rewrite the ith component of $K_t(u)$ in the form

$$(K_t(u))_i = -\sum_{j=1}^{n} P\nabla_j e^{tA}u_j u_i. \tag{4.5}$$

We now prove the following result.

THEOREM 4.1. *Let $K_t(u)$ be defined by* (4.5). *Then for each $u_0 \in E_p$ there exists a $T > 0$ such that* (4.4) *has a unique local solution $u \in C([0,T);\ E_p)$ provided $p > n$.*

PROOF. We verify the hypothesis of Theorem 1.1. Since e^{tA} commutes with P and ∇_j, (a) follows immediately, while (c) is trivial since $K_t(0) = 0$. For (b), note that

$$\begin{aligned}(K_t(u))_i - (K_t(v))_i &= -\sum_{j=1}^{n} P\nabla_j e^{tA}(u_j u_i - v_j v_i) \\ &= -\sum_{j=1}^{n} P\nabla_j e^{tA}\left[(u_j - v_j)u_i + v_j(u_i - v_i)\right]\end{aligned} \tag{4.6}$$

for $u, v \in E_p$ so that, with K_1, C, and d as in §2, and $p = n(\gamma - 1)/m$,

(4.7)

$$\begin{aligned}\|K_t(u) - K_t(v)\|_p &\leqslant \sup_i \sum_{j=1}^{n} \left\|\nabla_j e^{tA}\left[(u_j - v_j)u_i + v_j(u_i - v_i)\right]\right\|_p \\ &\leqslant \sup_i \sum_{j=1}^{n} K_1 C(1 + dt^{-r})\left\|(u_j - v_j)u_i + v_j(u_i - v_i)\right\|_{p/2},\end{aligned}$$

where $r = (m+1)/2$. Applying Hölder's inequality to the right-hand side of (4.7), we have

(4.8)

$$\begin{aligned}&\|K_t(u) - K_t(v)\|_p \\ &\quad\leqslant \sup K_1 C(1 + dt^{-r})\left[\left(\sum_{j=1}^{n}\|u_j - v_j\|_p\right)\|u_i\|_p + \left(\sum_{j=1}^{n}\|v_j\|_p\right)\|u_i - v_i\|_p\right] \\ &\quad\leqslant K_1 C(1 + dt^{-r})\left[\alpha n\|u - v\|_p + n\alpha\|u - v\|_p\right] \\ &\quad\leqslant (2\alpha n K_1 C(1 + dt^{-r}))\|u - v\|_p,\end{aligned}$$

where we have chosen α such that $\|u\|_p$, $\|v\|_p \leqslant \alpha$. If $p > n$ then $r < 1$, so that part (b) of Theorem (1.1) is verified with $C_\alpha(t) = 2\alpha n K_1 C(1 + dt^{-r})$. This concludes the proof of Theorem 4.1.

5. The generalized Burgers' equation in bounded domains. If we replace $\mathbf{R}^n$ by a bounded domain Ω with smooth boundary, then for most commonly occurring boundary conditions, e.g. $u = 0$ on $\partial\Omega$, it is no longer true that Δ commutes with $\operatorname{div}(\cdot)$. In this section, therefore, we modify the somewhat more complex arguments of Theorem 2 of [**11**], which were designed to handle the Navier-Stokes equation in this setting. We first note some facts about $e^{t\Delta}$ with $D_p(\Delta) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, where $1 \leqslant p < +\infty$.

By Proposition 3.1 of [**13**], given $T > 0$ there exists a constant C_1 such that for all $f \in L^p(\Omega)$

$$\|e^{t\Delta} f\|_q \leqslant C_1 t^{-n/(2r)} \|f\|_p \tag{5.1}$$

whenever $1 \leqslant p < q < +\infty$ and $t \in (0, T]$; here $1/r = 1/p - 1/q$, and C_1 depends on p, q, and Ω. Moreover for all $f \in L^p(\Omega)$

$$\lim_{t \downarrow 0} t^{n/(2r)} \|e^{t\Delta} f\|_q = 0. \tag{5.2}$$

Meanwhile, for $1 < p < +\infty$ there exists a constant C_2, depending only on p and Ω, such that

$$\|e^{t\Delta} f\|_{1,p} \leqslant C_2 t^{-1/2} \|f\|_p \tag{5.3}$$

for all $f \in L^p(\Omega)$. This follows from the analyticity of $e^{t\Delta}$. By using the techniques of [**11** or **13**] we also have, for all $f \in L^p(\Omega)$, that

$$\lim_{t \downarrow 0} t^{1/2} \|e^{t\Delta} f\|_{1,p} = 0. \tag{5.4}$$

We rewrite (2.1) in the form

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} g(u(s)) \cdot \nabla u(s)\, ds, \tag{5.5}$$

where $g = \psi'$. Note that the maximum degree of the components of g is $\gamma - 1$; for what follows set $\gamma - 1 = \theta$.

THEOREM 5.1. *Let p satisfy $p \geqslant \theta n$ with $n \geqslant 2$. Then there exists a $T > 0$ such that* (5.5) *has a unique solution $u \in C([0, T);\ L^p(\Omega))$.*

PROOF. We mimic the proof of Theorem 2 of [**11**] and select $W = L^{p/2}(\Omega)$, $X = L^p(\Omega)$, $Y = L^{p\theta}(\Omega)$, and $Z = W^{1,p}(\Omega)$. For simplicity we suppose the components of g are monomials of degree θ, and that $\theta > 1$. The case $\theta = 1$ is clearly already handled by Theorem 2 of [**11**] since $g(u) \cdot \nabla u$ is the diagonal part of a bilinear form for $\theta = 1$. Fix $u_0 \in X$ and let α, β, T be such that $\beta \to 0$ as $T \to 0$ and

$$\|e^{tA} u_0\|_X \leqslant \alpha, \quad t^b \|e^{t\Delta} u_0\|_Y \leqslant \beta, \quad t^{1/2} \|e^{t\Delta} u_0\|_Z \leqslant \beta \tag{5.6}$$

for all $t \in (0, T]$, where $b = n(\theta - 1)/(2p\theta)$. Note that the latter two inequalities follow from (5.1)–(5.4).

For α, β, T as above, let M be the space of all curves u: $[0, T] \to X$ such that

(i) u: $[0, T] \to X$ is continuous and $\|u(t)\|_X \leqslant 2\alpha$ for $t \in [0, T]$;

(ii) u: $(0, T] \to Y$ is continuous and $t^b\|u(t)\|_Y \leqslant 2\beta$ for $t \in (0, T]$;

(iii) u: $(0, T] \to Z$ is continuous and $t^{1/2}\|u(t)\|_Z \leqslant 2\beta$ for $t \in (0, T]$.

M is a nonempty complete metric space with metric ρ where, for $u, v \in M$,

$$\rho(u, v) = \sup_{t \in (0, T]} \left\{ \|u(t) - v(t)\|_X, t^b\|u(t) - v(t)\|_Y, t^{1/2}\|u(t) - v(t)\|_Z \right\}.$$

We want to be able to select T such that the map

$$(Su)(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}G(u(s))\,ds \tag{5.7}$$

is a strict contraction on M, where $G(u) = g(u) \cdot \nabla u$. As a preliminary estimate, note that there exists a constant C_3 depending on g and n such that

$$\begin{aligned} \|G(u) - G(v)\|_W &\leqslant \|g(u) - g(v)\|_p\|\nabla u\|_p + \|g(v)\|_p\|\nabla(u - v)\|_p \\ &\leqslant C_3\left[\left(\sum_{k=0}^{\theta-1} \|u\|_{p\theta}^{\theta-1-k}\|v\|_{p\theta}^k\right)\|u - v\|_{p\theta}\|\nabla u\|_p + \|v\|_{p\theta}^\theta\|\nabla(u - v)\|_p\right], \end{aligned} \tag{5.8}$$

where we have used Hölder's inequality and the fact that

$$u^\theta - v^\theta = \left(\sum_{k=0}^{\theta-1} u^{\theta-1-k}v^k\right)(u - v).$$

Hence if $u, v \in M$

$$\begin{aligned} \|G(u(t)) - G(v(t))\|_W &\leqslant C_3\left[\theta(2\beta)^{\theta-1}t^{-t(\theta-1)}(2\beta)t^{-1/2}t^{-b}\rho(u, v) + (2\beta)^\theta t^{-b\theta}t^{-1/2}\rho(u, v)\right] \\ &= C_3(\theta + 1)(2\beta)^\theta t^{-(\theta b+1/2)}\rho(u, v) \\ &\equiv C_4 t^{-(\theta b+1/2)}(2\beta)^\theta\rho(u, v) \end{aligned} \tag{5.9}$$

so that (noting that $e^{t\Delta} = e^{(t/2)\Delta}(e^{(t/2)\Delta})$)

$$\begin{aligned} t^{1/2}\|(Su)(t) - (Sv)(t)\|_Z &\leqslant t^{1/2}\int_0^t C_2(2)^{-1/2}(t - s)^{-1/2}\|e^{[(t-s)/2]\Delta}(G(u(s)) - G(v(s)))\|_p\,ds \\ &\leqslant t^{1/2}\int_0^t C_2(2)^{-(1/2+a)}(t - s)^{-(1/2+a)}\|G(u(s)) - G(v(s))\|_W\,ds \\ &\leqslant t^{1/2}C_5\left[\int_0^t (t - s)^{-(1/2+a)}s^{-(\theta b+1/2)}\,ds\right](2\beta)^\theta\rho(u, v), \end{aligned} \tag{5.10}$$

where $C_5 = (C_2/2)C_4$ and $a = n/(2p)$. Similarly

$$\|S(u(t)) - S(v(t))\|_X \leqslant C_6\left[\int_0^T (t - s)^{-a}s^{-(\theta b+1/2)}\,ds\right](2\beta)^\theta\rho(u, v) \tag{5.11}$$

and

(5.12)

$$\|S(u(t)) - S(v(t))\|_\gamma \leqslant C_7 t^b \left[\int_0^t (t-s)^{-(b+a)} s^{-(\theta b+1/2)} ds\right] (2\beta)^\theta \rho(u,v)$$

for appropriate constants C_6 and C_7.

Now, if $c, d \in (0,1)$, a simple scaling argument (see [**13**]) shows that

$$\int_0^t (t-s)^{-c} s^{-d} ds = t^{1-c-d} \int_0^1 (1-s)^{-c} s^{-d} ds \tag{5.13}$$

so that the right-hand side of (5.10) is

$$C_5 t^{1-(a+\theta b+1/2)} \left[\int_0^1 (1-s)^{-(1/2+a)} s^{-(\theta b+1/2)} ds\right] (2\beta)^\theta \rho(u,v) \tag{5.14}$$

and the right-hand sides of (5.11) and (5.12) can be similarly rewritten. Hence if $a + \theta b + 1/2 \leqslant 1$, which implies $p \geqslant \theta n$, estimates (5.10)–(5.12) imply that S is a strict contraction from M to M for small enough $T > 0$, since β can be chosen sufficiently small if T is small enough. This concludes the proof of Theorem 5.1, provided we note that calculations similar to those above show that S maps M to M for T sufficiently small.

Meanwhile, since $u(t) \in W_{1,p}(\Omega)$ for $t > 0$ with $p > n$, then $u(t) \in C_B(\Omega) \subset L^\infty(\Omega)$ for $t > 0$. The following is a corollary of the preceding result and the proof of Theorem 3.1.

THEOREM 5.2. *The solution u found in Theorem* 5.1 *is a global solution when* $p > n$, *i.e.*, $u \in C([0, +\infty); L^p(\Omega))$.

Finally, we remark that we have proven Theorem 5.1 when the components of ϕ are monomials of degree θ. The general polynomial case can be obtained via a more careful analysis based on more complex versions of (5.8), using the fact that polynomials of degree θ give rise to locally Lipschitz maps from $L^p(\Omega)$ to $L^{p\theta}(\Omega)$ when Ω is bounded. We also note that we have not handled the case $n = 1$, but the above proof will go through in that case if we additionally require that $p \geqslant 2$.

6. Remarks. We note that an equation similar to (1.3) was considered in [**9**], where local existence and uniqueness for mild solutions was established for initial data in $L^4(\Omega)$ when $n = 3$ and $\gamma = 2$. In §2 we have improved this to (1.4), which reduces to $p > 3$ in the above case, and in §5 we have noted that a modification of the theory of [**11**] extends (1.4) further to allow $p = (\gamma - 1)n$.

Meanwhile the arguments of §3 apply to both §§2 and 5 to obtain global existence when (1.4) is satisfied. Thus the blowup behavior of (1.1) studied by Giga and Weissler (see e.g., [**7**, **8**, and **12**]) does not appear in a similar fashion for (1.3). This is not particularly surprising since it is intuitively reasonable that the maximum principle can be applied in some way (see [**2**]).

Finally, we note that the main difference between the Navier-Stokes equation and (1.3) is the projection operator P. It is well known that the nonlocal nature of P has so far prevented the establishment of anything like the maximum principle for (1.5).

It is evident from §§4 and 5, however, that the local existence theory of (1.3) and (1.5) is very similar, which is in large part due to the fact that P is a bounded map from L^p to L^p.

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