PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 1, September 1987

POINTWISE CONVERGENT APPROXIMATE IDENTITIES OF DILATED RADIALLY DECREASING KERNELS

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ABSTRACT. Let ϕ be integrable on R^n with $\int_{R^n} \phi(y)\,dy = 1$. It is shown that

$$\lim_{\varepsilon \to 0+} (\phi_\varepsilon^* f)(x) = \lim_{\varepsilon \to 0+} \varepsilon^{-n} \int_{R^n} \phi\left(\frac{x-y}{\varepsilon}\right) f(y)\,dy = f(x)$$

a.e. on R^n, whenever the least radially decreasing majorant of ϕ, defined by $\psi(x) = \sup_{|y| \geqslant |x|} |\phi(y)|$, is such that $|x|^n \psi(x) = |x|^n \psi(|x|)$ is nonincreasing in $|x|$ when $|x|$ is large and $(\psi_{\varepsilon_0}^* |f|)(x_0) < \infty$ for some $x_0 \in R^n$ and $\varepsilon_0 > 0$.

1. Introduction. A well-known result of Calderón and Zygmund [**1**, p. 111, Lemma 1] asserts that if the least radially decreasing majorant Ψ of a Lebesgue-measurable function ϕ on R^n, $\Psi(x) = \sup_{|y| \geqslant |x|} |\phi(y)|$, is integrable, then, for all $f \in L^p(R^n)$, $1 \leqslant p < \infty$,

$$\lim_{\varepsilon \to 0+} (\phi_\varepsilon^* f)(x) = \lim_{\varepsilon \to 0+} \varepsilon^{-n} \int_{R^n} \phi\left(\frac{x-y}{\varepsilon}\right) f(y)\,dy = f(x) \int_{R^n} \phi(y)\,dy \tag{1.1}$$

for a.e. $x \in R^n$. In this note we prove the simple additional requirement that $|x|^n \psi(x) = |x|^n \psi(|x|)$ be nonincreasing in $|x|$ for $|x|$ large guarantees (1.1) whenever $\psi_\varepsilon^* f$ is defined for small ε. Examples of such ϕ are the Poisson and Weierstrass kernels considered for R^1 in Hardy [**2**, **3**, **4**] (see also Titchmarsh [**6**, pp. 30–32] and Hirschman-Widder [**5**, p. 175, pp. 188–189]).

In §2 we prove our theorem and in §3 give an example to show that some restriction like the one put on ψ is necessary.

As usual, C_n will denote a positive dimensional constant that may change from line to line.

2. The theorem. We begin with a simple lemma that will ensure $\phi_\varepsilon^* f$ is defined when ε is small. As our theorem reduces to that of Calderón and Zygmund when ϕ has compact support (and f is locally integrable), we will assume $\psi > 0$.

LEMMA. *Suppose $\Psi > 0$ is a radially decreasing, integrable function on R^n such that*

$$|x|^n \Psi(|x|) \text{ is nonincreasing in } |x| \text{ for } |x| \text{ large}. \tag{2.1}$$

If $(\Psi_{\varepsilon_0}^* |f|)(x_0) < \infty$ for some $x_0 \in R^n$ and $\varepsilon_0 > 0$, then, for a.e. $x \in R^n$, $(\Psi_\varepsilon^* |f|)(x) < \infty$ whenever $0 < \varepsilon < \varepsilon_0$.

Received by the editors June 10, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 42B99; Secondary 44A35.
Research partly supported by NSERC grant A4021.

PROOF. We have

$$(\Psi_\varepsilon^*|f|)(x)=\int_{R^n}\Psi_\varepsilon(|x-y|)|f(y)|\,dy=\int_{|x-y|\leqslant M}+\int_{|x-y|>M}=I+J.$$

Now,

$$J=\int_{|x-y|>M}\psi_{\varepsilon_0}(|x_0-y|)|f(y)|g(y)\,dy,$$

where

$$g(y)=[r^n\Psi(r)/r_0^n\Psi(r_0)]\left[\frac{|x_0-y|}{|x-y|}\right]^n,\quad r=|x-y|/\varepsilon,\quad r_0=|x_0-y|/\varepsilon_0$$

is bounded, given $M>0$ sufficiently large and $0<\varepsilon<\varepsilon_0$, in view of (2.1).

Again, letting

$$E_k=\{y\in R^n\colon 2^k\varepsilon\leqslant|x-y|<2^{k+1}\varepsilon\},\qquad k=0,\pm1,\pm2,\ldots,$$

$$\begin{aligned}I&\leqslant\sum_{k=-\infty}^{\alpha(\varepsilon)}\int_{E_k}\Psi_\varepsilon(|x-y|)|f(y)|\,dy,\qquad \alpha(\varepsilon)=[\log_2 M/\varepsilon],\\ &\leqslant\sum_{k=-\infty}^{\alpha(\varepsilon)}\Psi_\varepsilon(2^k\varepsilon)\int_{|x-y|\leqslant 2^{k+1}\varepsilon}|f(y)|\,dy\\ &\leqslant 2^n\sum_{k=-\infty}^{\alpha(\varepsilon)}2^{kn}\psi(2^k)[2^{k+1}\varepsilon]^{-n}\int_{|x-y|\leqslant 2^{k+1}\varepsilon}|f(y)|\,dy\\ &\leqslant 2^n\left[\sup_{0<\alpha<2M}\alpha^{-n}\int_{|x-y|\leqslant\alpha}|f(y)|\,dy\right]\sum_{k=-\infty}^{\infty}2^{kn}\Psi(2^k),\end{aligned}$$

is finite for a.e. x (for example, x in the set of Lebesgue points of the locally-integrable function f), since Ψ is radially decreasing and integrable.

THEOREM. *Let ϕ be an integrable function of R^n with $\int_{R^n}\phi(x)\,dx=1$. Suppose that $\Psi(x)=\sup_{|y|\geqslant|x|}|\phi(y)|$, the least radially decreasing majorant of ϕ, is positive, integrable, and satisfies* (2.1) *when $|x|\geqslant r_0>0$. Then, given Lebesgue-measurable f on R^n, the averages $(\phi_\varepsilon^*f)(x)$ are defined at almost all $x\in R^n$ when ε is close to* 0 *and*

(2.2) $$\lim_{\varepsilon\to0+}\phi_\varepsilon^*f=f\quad a.e.$$

provided

(2.3) $$(\Psi_{\varepsilon_0}^*|f|)(x_0)<\infty$$

for some $x_0\in R^n$ and $\varepsilon_0>0$. In particular, if ϕ is radially decreasing so that $\phi=\Psi$, then (2.3) *is necessary and sufficient for* (2.2).

PROOF. To begin the proof we observe that (2.3) implies f is locally-integrable. This means, by the Lemma, that ϕ_ε^*f is defined at all $x\in L_f$, the Lebesgue set of f, when $0<\varepsilon<\varepsilon_0$. Now, fixing a positive $\varepsilon<\varepsilon_0$ and $x\in L_f$,

$$\begin{aligned}|(\phi_\varepsilon^*f)(x)-f(x)|&\leqslant\int_{R^n}\Psi_\varepsilon(|x-y|)|f(y)-f(x)|\,dy\\ &\leqslant\int_{|x-y|\leqslant1}+\int_{|x-y|>1}=I_\varepsilon+J_\varepsilon.\end{aligned}$$

Arguing as in the Lemma, we obtain

$$I_\varepsilon \leqslant 2^n \sum_{k=-\infty}^{\alpha(\varepsilon)} 2^{kn}\Psi(2k)[2^{k+1}\varepsilon]^{-n}\int_{|x-y|\leqslant 2^{k+1}\varepsilon} |f(y)-f(x)|\,dy,$$

with $\alpha(\varepsilon) = [\log_2 1/\varepsilon]$. Hence, $\lim_{\varepsilon\to 0+} I_\varepsilon = 0$, whenever $x \in L_f$, by the dominated convergence theorem, since $\sum_{k=-\infty}^{\infty} 2^{kn}\Psi(2^k) < \infty$.

As

$$\int_{|x-y|\geqslant a} \Psi_\varepsilon(|x-y|)\,dy = \int_{|z|\geqslant a/\varepsilon} \Psi(|z|)\,dz \to 0 \tag{2.4}$$

when $\varepsilon \to 0+$ for each fixed $a > 0$, it suffices to show

$$\lim_{\varepsilon\to 0+} \int_{|x-y|>1} \Psi_\varepsilon(|x-y|)|f(y)|\,dy = 0 \tag{2.5}$$

in order to obtain $\lim_{\varepsilon\to 0+} J_\varepsilon = 0$ But,

$$\Psi_\varepsilon(y) \leqslant C_n|y|^{-n}\int_{|y|/2\leqslant|z|\leqslant|y|} \Psi_\varepsilon(|z|)\,dz \to 0, \qquad R^n \ni y \neq 0,$$

by (2.4), so (2.5) follows by dominated convergence since

$$\begin{aligned}\Psi_\varepsilon(|x-y|)|f(y)| &= \left[|x-y|/\varepsilon\right]^n\Psi(|x-y|/\varepsilon)|f(y)|/|x-y|^n\\ &\leqslant C_n\Psi_\delta(|x-y|)|f(y)|\end{aligned}$$

when $0 < \varepsilon \leqslant \delta = \min[\varepsilon_0/2, r_0^{-1}]$.

REMARKS. 1. Essentially the same proof shows that when μ is a Borel measure on R^n, with total variation, $|\mu|$, finite on compact sets,

$$\lim_{\varepsilon\to 0+} \phi_\varepsilon^*\mu = D\mu,$$

the Radon-Nikodým derivative of μ, a.e., provided $(\Psi_{\varepsilon_0}^*|\mu|)(x_0) < \infty$ for some $x_0 \in R^n$ and $\varepsilon_0 > 0$. Here, of course,

$$(\phi_\varepsilon^*\mu)(x) = \varepsilon^{-n}\int_{R^n} \phi\left(\frac{x-y}{\varepsilon}\right) d\mu(y).$$

2. Easy modifications of the argument of the Theorem yield (2.2) when condition (2.1) is replaced by $f \in L^\infty(R^n)$ or $\int_{R^n}(|f(y)|/(1+|y|^n))\,dy < \infty$. The latter condition holds, for example, if $f \in L^p(R^n)$ for some p, $1 \leqslant p < \infty$.

3. The example. We show that in the absence of restrictions on f like those in Remark 2 above, a condition on ϕ such as (2.1) is necessary. To this end, we construct nonnegative, locally-integrable functions f and ϕ on R^n, ϕ radially decreasing and integrable, such that 0 is a Lebesgue point of f and

$$\sup_{0<\varepsilon<1} (\phi_\varepsilon^* f)(0) < \infty, \tag{3.1}$$

yet

$$\lim_{\varepsilon\to 0+} (\phi_\varepsilon^* f)(0) \neq f(0)\int_{R^n} \phi(y)\,dy. \tag{3.2}$$

These functions will be constant on the annuli

$$\{y \in R^n \colon 2^k \leqslant |y| < 2^{k+1}\},$$

say $f = f_k$ and $\phi = \phi_k$, $k = 0, 1, 2, \ldots$, with $f = 0$, $\phi = \phi_0$, when $|y| < 1$.

It is enough to consider the sequence $\{\varepsilon_j\}$, $\varepsilon_j = 2^{-j}$, $j = 0, 1, \ldots$, instead of the continuous variable ε. In this case, letting $a_k = 2^{nk}\phi_k$, (3.1) and (3.2) amount to

$$\sup_{j \geqslant 0} \sum_{k=0}^{\infty} a_{k+j} f_k < \infty \tag{3.1$'$}$$

and

$$\lim_{j \to \infty} \sum_{k=0}^{\infty} a_{k+j} f_k \neq 0. \tag{3.2$'$}$$

Since (2.1) is not to hold, we do not want the sequence $\{a_k\}$ nonincreasing; however, we do want $\{\phi_k\} = \{2^{-nk} a_k\}$ nonincreasing and $\int_{R^n} \phi(x)\, dx \approx \sum_{k=0}^{\infty} a_k < \infty$. Thus, we define

$$\begin{aligned}
&a_0 = 1/3^n, \quad a_1 = 2^n/3^n, \\
&a_2 = 1/3^{2n}, \quad a_3 = 2^n/3^{2n}, \quad a_4 = 2^{2n}/3^{2n}, \\
&a_5 = 1/3^{3n}, \quad a_6 = 2^n/3^{3n}, \quad a_7 = 2^{2n}/3^{3n}, \quad a_8 = 2^{3n}/3^{3n}
\end{aligned}$$

and so on. We take $f_2 = 1/a_4$, $f_5 = 1/a_8, \ldots$ and $f_k = 0$ for all other k. This ensures the sum in (3.2)$'$ is greater than 1 for all $j = 2, 3, \ldots$.

It only remains to verify (3.1)$'$. Fix j. Let $k(1) = 0$, $k(2) = 2$, $k(3) = 5$, $k(4) = 9, \ldots$. Then,

$$\sum_{k=0}^{\infty} a_{k+j} f_k = \sum_{i=1}^{\infty} a_{k(i)+j} f_{k(i)}. \tag{3.3}$$

Now, when $j \geqslant 2$,

$$\sum_{i=j}^{\infty} a_{k(i)+j} f_{k(i)} = 1 + 1/2^n + 1/2^{2n} + \cdots = 2^n/(2^m - 1).$$

As for the first $j - 1$ terms on the right side of (3.3), they are $j - 1$ distinct terms in the convergent series $\sum_{m=1}^{\infty} \sum_{l=-\infty}^{m} 2^{nl}/3^{nm}$.

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