REGULARIZING PROPERTIES OF NONLINEAR SEMIGROUPS

SEMION GUTMAN

ABSTRACT. It is known that some classes of \( m \)-accretive operators \( A \) generate Lipschitz continuous semigroups of contractions; that is \( \| S(t + h)x - S(t)x \| \leq L(\| x \|)h/t, 0 \leq t \leq t + h \leq T, x \in D(A) \). If the underlying Banach spaces \( X \) and \( X^* \) are uniformly convex and an \( m \)-accretive operator \( B \) is bounded, we prove, in particular, that the semigroup generated by \( A + B \) is Hölder continuous. The proof is based on a result on the structure of accretive operators obtained via the Kuratowski-Ryll-Nardzewski Selection Theorem. Also, we consider some applications of these results to the existence of solutions of \( u' + Au + Bu = Cu, u(0) = u_0 \).

Let \( X \) be a Banach space with the norm \( \| \cdot \| \). The duality mapping \( J \) from \( X \) into the class of subsets of its dual \( X^* \) is defined by \( J(x) = \{ z \in X^*: z(x) = \| x \|^2 = \| z \|^2 \} \) for each \( x \in X \). For each pair \( x, y \in X \) we define

\[
\langle y, x \rangle_s = \sup \{ z(y) : z \in J(x) \}.
\]

A (multivalued) operator \( A \subset X \times X \) is called accretive if \( \langle y_1 - y_2, x_1 - x_2 \rangle_s \geq 0 \) for any \( (x_1, y_1), (x_2, y_2) \in A \). If the range \( R(I + \lambda A) = X \) for some (equivalently for all) \( \lambda > 0 \), then the operator \( A \) is called \( m \)-accretive. (General works on this subject are, e.g., [1, 5, 7].)

Crandall's and Liggett's theorem [5] states that any \( m \)-accretive operator \( A \) generates a (nonlinear) semigroup of contractions \( S(t): D(A) \rightarrow D(A), t \geq 0, \) by \( S(t)x = \lim_{n \rightarrow \infty} [(I + t/n)A]^{-n}x, x \in D(A) \). The function \( u(t) = S(t)u_0 \) can be considered as a generalized solution of the initial value problem

\[
\begin{aligned}
u'(t) + Au(t) &\geq 0, \quad t \geq 0, \\
u(0) &= u_0 \in D(A)
\end{aligned}
\]

(see, e.g., [1, §3.1; 5]).

Generally, little can be said on the regularity of the function \( u(t) \). In particular, \( u \) can be nondifferentiable, or, even, not Lipschitz continuous. However, for some classes of \( m \)-accretive operators the generated semigroup \( S(t) \) behaves better. For example, if \( S(t) \) is generated by the subdifferential of a convex, lower-semicontinuous function \( \phi(x) \) on a Hilbert space \( X \), then \( \| S(t + h)x - S(t)x \| \leq L(\| x \|)h/t, 0 \leq t \leq t + h \leq T, x \in D(A) \); where \( T > 0 \) and \( L(r) \) is a nondecreasing function [3; 1, §4.2].

Received by the editors March 28, 1985 and, in revised form, June 4, 1986.
Key words and phrases. Nonlinear semigroup, regularity, \( m \)-accretive, evolution equation, bounded perturbation.
The semigroups generated by homogeneous operators have similar behavior (see [4, Theorem 1]).

The main purpose of this note is to study such regularizing properties for the solutions of the perturbed initial value problem

$$\begin{cases} u'(t) + Au(t) + Bu(t) \equiv 0, \\ u(0) = u_0, \quad 0 \leq t \leq T, \end{cases}$$

where $A$ is an $m$-accretive operator and $B: X \to 2^X$ is a bounded (multivalued) operator. ($B$ is bounded if $\sup\{\|y\|: y \in Bx, \|x\| \leq r\}$ is finite for any $r > 0$.) To define a solution of (2) we recall [2; 1, 3.2] that for any integrable function $f \in L^1(0, T; X)$, $T > 0$, there exists a unique integral solution of the initial value problem

$$\begin{cases} u'(t) + Au(t) \equiv f(t), \\ u(0) = u_0. \end{cases}$$

That is a continuous function $u: [0, T] \to X$ such that

$$\frac{1}{2}\|u(t) - x\|^2 \leq \frac{1}{2}\|u(s) - x\|^2 + \int_s^t \langle f(\tau) - y, u(\tau) - x \rangle, d\tau$$

for any $0 \leq s \leq t \leq T$ and $[x, y] \in A$. A continuous function $u$ is called the strong solution of (3) if $u$ is continuous, differentiable a.e. on $[0, T]$, and (3) is satisfied a.e. on $[0, T]$ (we use only the Lebesgue measure on $[0, T]$).

We define the operator $H: L^1(0, T; X) \to C([0, T]; X)$ by $Hf = u$, where $u$ is the unique integral solution of (3). It is known [7, Remark 8.4] that

$$\|Hf(t) - Hg(t)\| \leq \|Hf(s) - Hg(s)\| + \int_s^t \|f(\tau) - g(\tau)\| d\tau$$

for $0 \leq s \leq t \leq T$.

**Definition 1.** A continuous function $u: [0, T] \to D(A)$ is called a solution of (2) if there exists a measurable function $f \in L^1(0, T; X)$ such that $f(t) \in -Bu(t)$ for a.e. $t \in [0, T]$ and $Hf = u$.

**Theorem.** Let $X$ be a Banach space, $T > 0$, $A$ be an $m$-accretive operator which generates a continuous semigroup of contractions $S(t)$, $t \geq 0$. Suppose that a (multivalued) operator $B: X \to 2^X$ is bounded and $D(B) \subset D(A)$. If there exists $\gamma > 0$, $0 < \mu \leq 1$, and a nondecreasing function $L: [0, \infty) \to [0, \infty)$ such that $\|S(t + h)x - S(t)x\| \leq t^{-\gamma}L(\|x\|)h^\mu$ for any $x \in D(A)$, $0 \leq t \leq t + h \leq T$, then any solution of (2) satisfies

$$\|u(t + h) - u(t)\| \leq C(\mu, \gamma, T, M, N)h^{\mu/(\gamma + 1)}, \quad 0 \leq t \leq t + h \leq T,$$

where $N = \max\{\|u(\tau)\|, 0 \leq \tau \leq T\}$ and $M = \sup\{\|y\|: y \in Bx, \|x\| \leq N\}$.

This Theorem follows immediately from the above definition and the following lemma.

**Lemma 1.** Suppose that $f \in L^1(0, T; X)$ and $\sup_{0 \leq \tau \leq T}\|f(\tau)\| \leq M$. If the operator $A$ satisfies the conditions of the Theorem then the integral solution $u(t)$ of (3) satisfies

$$\|u(t + h) - u(t)\| \leq C(\mu, \gamma, T, M, N)h^{\mu/(\gamma + 1)}$$

where the constants are as in the Theorem.
Proof. Fix \( t > 0 \). For any \( 0 \leq h \leq T - t \) and \( 0 \leq \sigma < t \) we have (by (5))

\[
\|u(t + h) - u(t)\| \leq \|u(t + h) - S(t - \sigma + h)u(\sigma)\| + \|S(t - \sigma + h)u(\sigma) - S(t - \sigma)u(\sigma)\|
\]

\[
\leq M(t - \sigma + h) + (t - \sigma)L\left(\|u(\sigma)\|\right) + M(t - \sigma).
\]

Thus \( \|u(t + h) - u(t)\| \leq \min_{0 \leq \sigma < t}[2M + 2M\sigma + L(N)(t - \sigma)^{-\gamma}h^\mu]. \)

Therefore

\[
\|u(t + h) - u(t)\| \leq h^{\mu/(\gamma + 1)}\left[2Mh^{1-\mu/(\gamma + 1)} + 2M\right] + L(N)^{(1/(\gamma + 1))(\gamma + 1)}\gamma^{-\gamma/(\gamma + 1)}
\]

\[
\leq h^{\mu/(\gamma + 1)}C_1(\mu, \gamma, T, M, N)
\]

for \( 0 < h < ((2M/\gamma L(N))t)\gamma+1)^{1/\mu} \) and

\[
\|u(t + h) - u(t)\| \leq t^{-\gamma}\left[2Mt^{\gamma+1} + Mht^{\gamma} + L(N)h^\mu\right]
\]

\[
\leq t^{-\gamma}\left[L(N)(\gamma + 1)h^\mu + Mht^\gamma\right] \leq t^{-\gamma}h^\mu/(\gamma + 1)C_2(\mu, \gamma, T, M, N)
\]

for \( h \geq ((2M/\gamma L(N))t^{\gamma+1})^{1/\mu} \). Thus

\[
\|u(t + h) - u(t)\| \leq h^{\mu/(\gamma + 1)}C_1 + t^{-\gamma}h^\mu/(\gamma + 1)C_2
\]

\[
\leq t^{-\gamma}h^\mu/(\gamma + 1)C(\mu, \gamma, T, M, N)
\]

and the proof of the lemma is complete.

It was assumed in the Theorem that the initial value problem (2) has a solution on \([0, T]\). It is known (see, e.g., [1, 3.3]) that such a solution exists for different assumptions on the operators \( A, B \) and the underlying Banach space \( X \). If the operator \( A \) is homogeneous and \( m \)-accretive and \( B \) is Lipschitz continuous, a stronger regularity result can be found in [4]. We are interested in the case where both operators \( A \) and \( B \) are \( m \)-accretive and not necessarily continuous. In the sequel we suppose that both \( X \) and \( X^* \) are uniformly convex and separable. Following [1, Chapters II and III] we define \( |Ax| = \inf\{\|y\|: y \in Ax\} \) and \( A^0x = \{y \in Ax: \|y\| = |Ax|\} \). It is known [1, p. 118] that in this case the operator \( A^0 \) is single-valued and \((d/dt)S(t)u_0 = A^0S(t)u_0, \) a.e. \( t > 0, u_0 \in D(A) \).

If we suppose that \( D(A) \subset D(B), \) and \( B \) is bounded, then [1, Theorem 2.3.5] the operator \( A + B \) is \( m \)-accretive. We denote the correspondent semigroup by \( S_{A+B}(t). \)

The following lemma handles the structure of the operator \( A + B \) via the Kuratowski-Ryll-Nardzewski Selection Theorem (see the appendix).

Lemma 2. Let \( X, X^* \) be uniformly convex, separable Banach spaces and let \( T > 0. \) Let operators \( B, A \) be \( m \)-accretive, \( B \) be bounded, and \( D(B) \supset D(A) \). Then \( u(t) = S_{A+B}(t)u_0, \) \( t \geq 0, u_0 \in D(A), \) is a solution of the system \( u' + Au + Bu = 0, u(0) = u_0, \) in the sense of Definition 1.

Proof. We should prove that there exists an integrable function \( f \in L^1(0, T; X), f(t) \in -Bu(t) \) a.e. \( t \in [0, T] \) such that \( Hf = u. \) Note that \( |Au_0| < \infty \) since \( u_0 \in D(A) \). Therefore \( |(A + B)u_0| < \infty \) and \( u(t) \) is the strong solution, that is \( u(t) \) exists for almost every \( t \in [0, T] \) and \( u'(t) + Au + Bu(t) = 0 \) a.e. on \([0, T] \).
By \([1, \text{Theorem 3.1.6}]\) \(u'(t) = (A + B)^0 u(t)\) a.e. \(t \in [0, T]\). Since \((A + B)^0 z = \lim_{\lambda \to 0} (A + B)_\lambda z\) for each \(z \in D(A + B)\) \((A + B)_\lambda\) is the Yosida approximation; see \([1, \text{Proposition 2.3.6}]\) we get that \(u'(t)\) is integrable as the pointwise limit of the uniformly bounded continuous functions \((A + B)_\lambda u(t)\). By Lemma 3 in the Appendix there exist measurable functions \(g\) and \(f\) such that \(-g(t) + f(t) = u'(t), g(t) \in Au(t), -f(t) \in Bu(t)\) a.e. \(t \in [0, T]\). These functions are integrable, since they are bounded. Thus \(u'(t) = f(t)\) or \(u'(t) + Au(t) \equiv f(t)\) a.e. \(t \in [0, T]\), \(u(0) = u_0\). Since \(u(t)\) is the strong solution of the above system it is also its integral solution.

**Corollary 1.** Let \(X, X^*\) be uniformly convex, separable Banach spaces and \(T > 0\). Let the operators \(B\) and \(A\) be \(m\)-accretive and let \(B\) be bounded. If \(D(B) \supset D(A)\) and the semigroup \(S_A(t)\) satisfies \(\|S_A(t + h)x - S_A(t)x\| < t^{-\gamma}L(||x||)h^p\) for any \(x \in D(A), 0 \leq t \leq t + h \leq T (\gamma > 0, 0 < \mu < 1, L\) is nondecreasing), then the semigroup \(S_{A+B}(t)\) satisfies

\[
\|S_{A+B}(t+h)u_0 - S_{A+B}(t)u_0\| \leq Ct^{-\gamma}h^{\mu/(\gamma+1)},
\]

where \(0 \leq t \leq t + h \leq T, u_0 \in D(A)\), and the constant \(C\) is chosen as in the Theorem.

**Proof.** It is enough to prove the assertion for any \(u_0 \in D(A)\). By Lemma 2 \(u(t) = S_{A+B}(t)u_0\) is the solution of (2) in the sense of Definition 1. By the Theorem it satisfies the required inequality.

**Remark.** A detailed analysis of the proof shows that it is enough to require \(D(A) \subset D(B)\) in the above corollary.

**Corollary 2.** Let \(X, X^*\) be uniformly convex, separable Banach spaces, \(A\) be a homogeneous \(m\)-accretive operator (see \([4]\)), and \(B\) be a bounded, everywhere defined in \(X\) \(m\)-accretive operator. Suppose that \(C: X \to X\) is a compact continuous operator. Then the initial value problem \(u'(t) + Au(t) + Bu(t) \equiv Cu(t), 0 \leq t \leq T, u(0) = u_0 \in D(A) \subset X\), has a local solution.

**Proof.** Since \(A\) is homogeneous the generated semigroup \(S_A(t), t \geq 0\), satisfies \(\|S(t+h)x - S(t)x\| \leq L(||x||)t^{-1}h\) for any \(x \in D(A), 0 \leq t \leq t + h \leq T\) (see \([4, \text{Theorem 1}]\)). By Corollary 1 the semigroup \(S_{A+B}(t)\) satisfies

\[
\|S_{A+B}(t+h)u_0 - S_{A+B}(t)u_0\| \leq Ct^{-1}h^{1/2}, \quad u_0 \in D(A).
\]

Therefore this semigroup \(S_{A+B}\) is equicontinuous (that is the family of functions \(\{t \mapsto S_{A+B}(t)x, x \in D(A + B) \cap G\}\) is equicontinuous on \((0, T)\) for any bounded subset \(G \subset X\)). By \([8, \text{Proposition 1.2}]\) the initial value problem has a local solution.

**Appendix.** We will use the Kuratowski-Ryll-Nardzewski Selection Theorem, which we will restate as follows:

**Theorem** \([6, \text{p. 286}]\). Let \(Y\) be a complete, separable metric space. Suppose \(F: [0, T] \to 2^Y\) is such that \(F(t) \subset Y\) is closed for any \(t \in [0, T]\), and for each open set \(V\) in \(Y\) the set \(\{\omega \in [0, T]: F(\omega) \cap V \neq \emptyset\}\) is measurable. Then there exists a measurable function \(f: [0, T] \to Y\) such that \(f(\omega) \in F(\omega)\) for each \(\omega \in [0, T]\).
Recall that if a separable Banach space $X$ and its dual $X^*$ are uniformly convex, then any convex, bounded, and closed subset $W \subset X$ is weakly compact [9, §5.2]. The set $W$ with the weak topology is a complete compact metric space.

**Lemma 3.** Let $X, X^*$ be uniformly convex and separable Banach spaces. Suppose that $T > 0$, $A, B$ are $m$-accretive operators in $X$, $\text{Dom}(A) \subseteq \text{Dom}(B)$, and $B$ is bounded. Let $u: [0, T] \to \text{Dom}(A)$ be continuous and $h: [0, T] \to X$ be measurable and bounded.

If $h(t) \in (A + B)u(t)$ almost everywhere on $[0, T]$ then there exist measurable functions $g$ and $f: [0, T] \to X$ such that $g(t) \in Au(t)$, $f(t) \in Bu(t)$, and $g(t) + f(t) = h(t)$ a.e. on $[0, T]$.

**Proof.** Define $F: [0, T] \to X \times X$ by $F(t) = \{(x, y) \in X \times X: x + y = h(t)$, $x \in Au(t)$, $y \in Bu(t)\}$. Note that $F(t) \neq \emptyset$ for any $t \in [0, T]$. Let $M_1 = \sup\{\|z\|: z \in Bu(t)$, $t \in [0, T]\}$, $M_2 = \sup\{\|h(t)\|: t \in [0, T]\}$, and $M = M_1 + M_2$. Then $\|x\|, \|y\| \leq M$ for any $(x, y) \in F(t)$, $0 \leq t \leq T$. Hence $F(t) \subseteq Y = Z \times Z$, where $Z = \{z \in X: \|z\| \leq M\}$. We supply $Z$ with the weak topology of $X$. Thus $Z$ is a compact (complete) metrizable space and so is $Y$ with an appropriate product topology. Let $Q \subset [0, T]$ be any closed subset such that $h|_Q$ is continuous. Let $W$ be any closed subset of $Y$. Then the set $\{\omega \in Q: F(\omega) \cap W \neq \emptyset\}$ is closed in $[0, T]$. Indeed, if $t_n \in Q$, $t_n \to t$ as $n \to \infty$ and $(x_n, y_n) \in F(t_n)$, then we can assume (passing to a subsequence) that $x_n \to x$, $y_n \to y$ in $Z$, since $Z$ is compact. That is $x_n \to x$, $y_n \to y$ weakly in $X$. Since the operators $A$ and $B$ are demiclosed [1, Proposition 2.3.5] we have $x \in Au(t)$ and $y \in Bu(t)$. Also $x + y = h(t)$ since for any functional $e \in X^*$ we get $e(x + y) = \lim_{n \to \infty} e(x_n + y_n) = \lim_{n \to \infty} e(h(t_n)) = e(h(t))$. Therefore the set $\{\omega \in [0, T]: F(\omega) \cap W \neq \emptyset\}$ is measurable in $[0, T]$ as the union of the above closed sets. Since any open set $V$ in $Y$ can be represented as a countable union of closed sets $W_n$ we get that $\{\omega \in [0, T]: F(\omega) \cap V \neq \emptyset\}$ is measurable in $[0, T]$ and the Kuratowski-Ryll-Nardzewski Selection Theorem can be applied to the multivalued function $F: [0, T] \to Y$ to obtain measurable functions $g(t)$ and $f(t)$ such that $g(t) + f(t) = h(t)$, $g(t) \in Au(f)$, and $f(t) \in Bu(t)$. The functions $g$ and $f$ are measurable as functions into $Z$. Since the Banach space $X$ is separable they are measurable as functions into $X$ with its strong topology by the Pettis theorem [9, §5.4]

**References**


Department of Mathematics, University of Oklahoma, Norman, Oklahoma 73019