PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 1, September 1987

K^n-POSITIVE MAPS IN C^*-ALGEBRAS

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ABSTRACT. Let K^n be the set of n-positive maps of $B(H)$ to $B(H)$. A K^n-positive map of a C^*-algebra A to $B(H)$ is a positive linear map ϕ such that $\sum \mathrm{Tr}(\phi(a_i)b_i^t) \geq 0$ for any $\sum a_i \otimes b_i \in \{x \in A \otimes_\gamma T(H) | K^n \ni^\forall \alpha, (\mathrm{id} \otimes \alpha)(x) \geq 0\}$. It is shown that the following three statements are equivalent. (1) Every K^n-positive map of A to $B(H)$ is K^{n+1}-positive. (2) Every K^n-positive map of A to $B(H)$ is completely positive. (3) A is an n-subhomogeneous C^*-algebra.

Introduction. The concepts of mapping cones K and K-positive maps which have introduced by Størmer in [**4**] are powerful tools when one considers the extension problem of positive maps in C^*-algebras.

In this paper, we investigate K^n-positive maps of a C^*-algebra A to the algebra $B(H)$ of all bounded operators on a Hilbert space H, which are induced by the intermediary of the set of all n-positive maps K^n (which is one of the mapping cones) in $B^2(H)^+$, and which may have a close relation with the extension problem. The positivity of K^n-positive maps is stronger than the positivity of $B^2(H)^+$-positive maps and weaker than the positivity of $CP(H)$-positive maps, and becomes stricter as n grows large.

For the definitions of mapping cones, $B^2(H)^+$, and $CP(H)$ we refer the reader to the paper [**4**].

According to Proposition 1 in the next section, a K^n-positive map is an n-positive map. This is not true, however, in the reverse direction, as it is known that a 1-positive map is not in general K^1-positive (that is $B^2(H)^+$-positive). The author has recently shown in [**2**] that the cone which corresponds to the n-positive maps of A to $B(H)$ is

$$C_n^A = \overline{\mathrm{Conv}}^\gamma \left\{ \left(\sum_i^n a_i \otimes b_i \right)^* \left(\sum_i^n a_i \otimes b_i \right) \middle| a_i \in A,\ b_i \in T(H),\ i = 1, \dots, n \right\}$$

in $(A \otimes_\gamma T(H))^+$, where $T(H)$ is the set of all trace class operators on H. Combined with the results of [**4**], the relation among the cones in $(A \otimes_\gamma T(H))^+$ is the following:

$$\begin{array}{ccccccc} C_1^A & \subset \cdots \subset & C_n^A & \subset & C_{n+1}^A & \subset \cdots \subset & (A \otimes_\gamma T(H))^+ \\ \cap & & \cap & & \cap & & \| \\ P(A, K^1) & \subset \cdots \subset & P(A, K^n) & \subset & P(A, K^{n+1}) & \subset \cdots \subset & P(A, CP(H)) \end{array}$$

When we study the extension problem of positive maps in C^*-algebras, it is important whether the above inclusions are strict or not. Thus, we encounter a problem which is similar to the conjecture posed by M. D. Choi [**1**] in 1972. He

Received by the editors January 21, 1986 and, in revised form, May 21, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L05.
Key words and phrases. n-positive map, K^n-positive map, completely positive map, n-subhomogeneous C^*-algebra.

0002-9939/87 $1.00 + $.25 per page

asked if every n-positive map of a C^*-algebra A to a C^*-algebra B is a completely positive map under the condition that every n-positive map of A to B is $(n+1)$-positive. (The answer to this was first given by J. Tomiyama [**5**] and independently by R. R. Smith [**3**], both in 1981.)

Now, our problem is as follows: If every K^n-positive map of a C^*-algebra A to $B(H)$ is a K^{n+1}-positive map, is every K^n-positive map of A to $B(H)$ a completely positive map?

The aim of this article is to show that the answer is affirmative.

Results. Throughout this paper, we assume that the Hilbert space H is infinite dimensional (not necessarily separable) if not otherwise stated and M_n is the $n \times n$ complex matrices. We denote by K^n the set of n-positive maps in $B(B(H), B(H))$. K^n is a mapping cone which is defined by Størmer in [**4**]. Let A be a C^*-algebra. We define a cone $P(A, K^n)$ in $(A \otimes_\gamma T(H))^+$ as follows:

$$P(A, K^n) = \left\{ x \in (A \otimes_\gamma T(H))^+ \middle| K^n \ni^\forall \alpha, (\mathrm{id} \otimes \alpha)(x) \geq 0 \text{ in } A \otimes_{\min} B(H) \right\}.$$

Let $\phi \in B(A, B(H))$. We say that ϕ is a K^n-positive map if $\tilde{\phi}$ is a bounded linear functional on $A \otimes_\gamma T(H)$ and positive on the cone $P(A, K^n)$, where

$$\tilde{\phi}\left(\sum a_i \otimes b_i\right) = \sum \mathrm{Tr}(\phi(a_i) b_i^t), \qquad a_i \in A,\ b_i \in T(H)$$

(b^t is the transposed element in $T(H)$ of b with respect to some fixed orthonormal basis).

The next proposition states a basic relation between n-positive maps and K^n-positive maps.

PROPOSITION 1. *Let A be a C^*-algebra. If ϕ is a K^n-positive map of A to $B(H)$, then ϕ is an n-positive map of A to $B(H)$.*

PROOF. It is sufficient to show that $P(A, K^n) \supset C_n^A$ by using the result in [**2**]. Take $x = (\sum_i^n a_i \otimes b_i)^* (\sum_i^n a_i \otimes b_i)$ in C_n^A, where $a_i \in A$ and $b_i \in T(H)$. For any α in K^n, $[\alpha(b_i^* b_j)]_{1 \leq i,j \leq n}$ is a positive operator in $B(H) \otimes M_n$. Then, there exist operators $\{c_{pi}\}_{1 \leq p,i \leq n}$ in $B(H)$ such that $[\alpha(b_i^* b_j)] = \sum_p^n [c_{pi}^* c_{pj}]$. Therefore, we have

$$\begin{aligned}(\mathrm{id} \otimes \alpha)\left(\left(\sum_i^n a_i \otimes b_i\right)^* \left(\sum_i^n a_i \otimes b_i\right)\right) &= \sum_{i,j}^n a_i^* a_j \otimes \alpha(b_i^* b_j) \\ &= \sum_p^n \left(\sum_i^n a_i \otimes c_{pi}\right)^* \left(\sum_j^n a_j \otimes c_{pj}\right) \geq 0.\end{aligned}$$

Hence, x is contained in $P(A, K^n)$.

The composition map of a K^n-positive map and a completely positive map is again K^n-positive. This is shown in the next proposition. Throughout the rest of this paper, the summation symbol $\sum$ denotes a finite sum without further mention.

PROPOSITION 2. *Let A and B be C^*-algebras. If ϕ is a completely positive map of A to B and ψ is a K^n-positive map of B to $B(H)$, then $\psi \circ \phi$ is a K^n-positive map of A to $B(H)$. (In this proposition, H need not be infinite dimensional.)*

PROOF. First we see that, for any $\sum a_i \otimes b_i$ in $P(A, K^n)$, $\sum \phi(a_i) \otimes b_i$ is in $P(B, K^n)$. In fact, for any $\alpha \in K^n$, we have that $\sum a_i \otimes \alpha(b_i)$ is positive in

$A \otimes_{\min} B(H)$. If e is a finite-dimensional projection in $B(H)$, $\sum a_i \otimes e\alpha(b_i)e$ is positive in $A \otimes M_{\dim(e)}$. Since ϕ is a completely positive map of A to B, $\sum \phi(a_i) \otimes e\alpha(b_i)e$ is positive in $B \otimes M_{\dim(e)}$. As the dimension of e is arbitrary, we conclude that $\sum \phi(a_i) \otimes \alpha(b_i)$ is positive in $B \otimes_{\min} B(H)$. Therefore, $\sum \phi(a_i) \otimes b_i$ is in $P(B, K^n)$. The point having been established, we now see that the K^n-positivity of ψ leads to the following inequality:

$$\widetilde{\psi \circ \phi}\left(\sum a_i \otimes b_i\right) = \sum \operatorname{Tr}(\psi \circ \phi(a_i) b_i) = \tilde{\psi}\left(\sum \phi(a_i) \otimes b_i\right) \geq 0.$$

This means that $\psi \circ \phi$ is K^n-positive.

We remark that Proposition 2 is valid for K-positive maps with respect to the arbitrary mapping cone K.

The next theorem is the most crucial part of this paper. We denote by K_m^n the set of n-positive maps of M_m to M_m. ϕ^* is the adjoint map of ϕ with respect to the inner product by the canonical trace Tr_m on M_m.

THEOREM 3. *If ϕ is an n-positive map of M_l to M_m $(l \leq m)$, then ϕ is a K_m^n-positive map.*

PROOF. First, we treat the case of $l = m$. We notice that, for an n-positive map ϕ of M_m to M_m, $t \circ \phi^* \circ t$ is n-positive again, where t is the transpose map on M_m. This is seen by the following calculation.

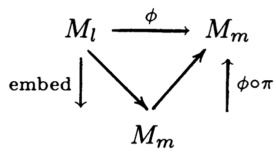
$$\begin{aligned}
&\operatorname{Tr}_m \otimes \operatorname{Tr}_n \left(\left(\sum_{i,j}^n t \circ \phi^* \circ t(a_i^* a_j) \otimes e_{ij} \right) \left(\sum_{p,q}^n b_p^* b_q \otimes e_{pq} \right) \right) \\
&= \operatorname{Tr}_m \left(\sum_{i,p}^n t \circ \phi^* \circ t(a_i^* a_p) b_p^* b_i \right) \\
&= \operatorname{Tr}_m \left(\sum_{i,p}^n \phi^*(a_p^t a_i^{t^*}) b_i^t b_p^{t^*} \right) \\
&= \operatorname{Tr}_m \left(\sum_{i,p}^n \phi(b_i^t b_p^{t^*}) a_p^t a_i^{t^*} \right) \\
&= \operatorname{Tr}_m \otimes \operatorname{Tr}_n \left(\left(\sum_{i,j}^n \phi(b_i^t b_j^{t^*}) \otimes e_{ij} \right) \left(\sum_{p,q}^n a_p^t a_q^{t^*} \otimes e_{pq} \right) \right) \\
&\geq 0 \quad \text{for any } a_i, b_i \in M_m \text{ and the matrix unit } \{e_{ij}\} \text{ of } M_n.
\end{aligned}$$

Therefore, for any $\sum a_i \otimes b_i \in P(M_m, K_m^n)$, we have that $\sum a_i \otimes t \circ \phi^* \circ t(b_i) \geq 0$ in $M_m \otimes M_m$. Thus, we conclude that

$$\begin{aligned}
\tilde{\phi}\left(\sum a_i \otimes b_i\right) &= \sum \operatorname{Tr}(\phi(a_i) b_i^t) = \sum \operatorname{Tr}(a_i^t t \circ \phi^* \circ t(b_i)) \\
&= \widetilde{\operatorname{id}}_m \left(\sum t \circ \phi^* \circ t(b_i) \otimes a_i \right) \geq 0.
\end{aligned}$$

Next we consider the case of $l \leq m$. Let π be a compression map of M_m to M_l such that $\pi(x) = exe$, where e is an l-dimensional projection. Then, $\phi \circ \pi$ is clearly

n-positive and moreover K_m^n-positive from the case treated. On the other hand, we can decompose ϕ as in the following diagram:



Hence, ϕ is a K_m^n-positive map by virtue of Proposition 2.

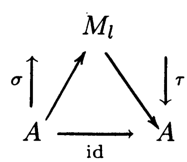
COROLLARY 4. *If $x \in M_l \otimes M_m$ $(l \leq m)$ satisfies the condition that $(\mathrm{id}_l \otimes \phi)(x) \geq 0$ for any n-positive map ϕ of M_m to M_m, then x is contained in the cone which is the closure of the convex hull of $(\sum_i^n a_i \otimes b_i)^*(\sum_i^n a_i \otimes b_i)$, where $a_i \in M_l$, $b_i \in M_m$.*

COROLLARY 5. *For any $l \in \mathbf{N}$, if ϕ is an n-positive map of M_l to $B(H)$, then it is a K^n-positive map.*

PROOF. We have only to prove that for any $x = \sum a_i \otimes b_i \in P(M_l, K^n)$, the inequality $\tilde{\phi}((1 \otimes e)x(1 \otimes e)) \geq 0$ holds for any finite-dimensional projection e such that $e^t = e$ and $\dim(e) \geq l$. Put $\psi(\cdot) = e\phi(\cdot)e$. If $\dim(e) = m$, then ψ is an n-positive map of M_l to M_m. Hence, ψ is K_m^n-positive. Since we can show easily that $(1 \otimes e)x(1 \otimes e)$ is contained in $P(M_l, K_m^n)$, we obtain the inequality

$$\tilde{\phi}((1 \otimes e)x(1 \otimes e)) = \sum \mathrm{Tr}(e\phi(a_i)eb_i^t e) = \tilde{\psi}((1 \otimes e)x(1 \otimes e)) \geq 0.$$

REMARK 6. It was pointed out to us by the referee that Corollary 5 is true for nuclear C^*-algebras and the improvement yields another proof of Størmer's theorem (Theorem 3.14 in [**4**]). In fact, suppose A is nuclear (there exist diagrams of completely positive contractions



which approximately commute in the point norm topology) and ϕ is an n-positive map of A to $B(H)$. Then $\phi \circ \tau$ is K^n-positive by Corollary 5 and $\phi \circ \tau \circ \sigma$ is also K^n-positive by Proposition 2. Since $\{\phi \circ \tau \circ \sigma\}$ converges to ϕ in the point norm topology, we have the K^n-positivity of ϕ.

Now, we show the main theorem.

THEOREM 7. *Let A be a C^*-algebra. Then the following three statements are equivalent.*

(1) *A is an n-subhomogeneous C^*-algebra.*
(2) *The cone $P(A, K^n)$ is equal to the cone $(A \otimes_\gamma T(H))^+$.*
(3) *The cone $P(A, K^n)$ is equal to the cone $P(A, K^{n+1})$.*

PROOF. (1) $\Rightarrow$ (2) Due to the fact which is proved in [**5**, **2**], we have

$$C_n^A = (A \otimes_\gamma T(H))^+.$$

Hence, Proposition 1 entails the inclusion $P(A, K^n) \supset (A \otimes_\gamma T(H))^+$. The reverse inclusion is trivial.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) If A is not n-subhomogeneous, in view of Lemma 1.1 of [**5**], there exist completely positive maps σ of A to M_{n+1} and ρ of M_{n+1} to A such that $\sigma \circ \rho$ is the identity map in M_{n+1}. Moreover, there exists an n-positive map ϕ of M_{n+1} to $B(H)$ which is not $(n+1)$-positive. ϕ is a K^n-positive map of M_{n+1} to $B(H)$ by Corollary 5. Hence, $\phi \circ \sigma$ is a K^n-positive map of A to $B(H)$ by Proposition 2. On the other hand, since ϕ is not $(n+1)$-positive, it is not K^{n+1}-positive by Proposition 1. By applying Proposition 2 to $\phi \circ \sigma$ and ρ again, we do not have that $\phi \cdot \sigma$ is K^{n+1}-positive. Hence, we obtain a K^n-positive map of A to $B(H)$ which is not K^{n+1}-positive. This means that $P(A, K^n) \subsetneq P(A, K^{n+1})$, which contradicts the assumed statement (3).

Now all implications are proved.

ACKNOWLEDGMENT. The author wishes to express his gratitude to Professor O. Takenouchi for various advice and constant encouragement and to Messrs. Y. Katayama, S. Kawakami, and M. Nagisa for fruitful discussions, and to the referee for helpful comments on Corollary 5.

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