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A THEOREM ON WEIGHTED L^1-APPROXIMATION

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ABSTRACT. It is proven that the A-property is necessary for a finite-dimensional subspace to be Chebyshev in $C(K)$ with respect to the weighted L^1-norm $\|f\|_w = \int_K w|f|\,d\mu$ for all weight functions w in certain classes of functions.

1. Introduction. Let K be a compact subset of $\mathbf{R}^s$ ($s \geq 1$) satisfying $K = \overline{\mathrm{Int}(K)}$ and denote by $C(K)$ the space of all continuous, real-valued functions on K. Let $W_\infty = \{w \in L^\infty(K)\colon w > 0 \text{ on } K\}$, and for $w \in W_\infty$ let $C_w(K)$ denote the space $C(K)$ with the w-weighted L^1-norm

$$\|f\|_w = \int_K w|f|\,d\mu, \tag{1}$$

where μ denotes Lebesgue measure. We say that a finite-dimensional subspace U of $C(K)$ is *Chebyshev* in $C_w(K)$ if every $f \in C(K)$ has a unique best approximation from U with respect to the norm (1).

Recent interest in the Chebyshev subspaces of $C(K)$ with L^1-norms was inspired by the discoveries that spaces of spline functions are Chebyshev in $C_1[0,1]$ in addition to the subspaces of $C[0,1]$ that satisfy the Haar condition on $(0,1)$ (see [**8**] and its references). A unifying feature of these spaces is the so called A-property (defined in §2), and Strauss [**10**] proved that if U satisfies the A-property, then U is Chebyshev in $C_1[0,1]$. This result is easily generalized for any $w \in W_\infty$ and any K as above. When $K = [0,1]$, Kroó [**1**] established a converse showing that if U is Chebyshev in $C_w[0,1]$ for all $w \in W_B = \{w \in W_\infty\colon \inf w > 0\}$, then U is an A-space, and Sommer [**7**] generalized this result to the multivariate setting. Independent of Kroó, Pinkus [**6**] sought a converse using only the continuous weight functions and succeeded under the additional assumption that $\mu(Z(u)) = \mu(\mathrm{Int}(Z(u)))$ for all $u \in U$ where $Z(u) = \{x \in K\colon u(x) = 0\}$. Subsequently, Kroó [**3**] removed this condition for any K as above showing that the A-property is necessary for U to be Chebyshev in $C_w(K)$ for all $w \in W_C = \{w \in C(K)\colon w > 0 \text{ on } K\}$. It is natural to ask whether the analytic or even the polynomial weight functions suffice. Indeed, when $K = [0,1]$ and U satisfied Pinkus' condition, Kroó [**2**] showed that the C^∞-weight functions suffice. In this note we give general conditions on $W \subseteq W_\infty$ so that the A-property is necessary for U to be Chebyshev in $C_w(K)$ for all $w \in W$.

THEOREM 1. *Let W be a conex cone in W_∞ satisfying the condition*

(2) *if q is a bounded, measurable function and $\int_K wq\,d\mu \geq 0$ for all $w \in W$, then $q \geq 0$ a.e. on K.*

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If a finite-dimensional subspace U of $C(K)$ is Chebyshev in $C_w(K)$ for all $w \in W$, then U is an A-space.

Evidently W_∞ satisfies condition (2) and an argument using Lusin's theorem shows that W_C also satisfies (2). Moreover any convex cone W in W_∞ whose L^∞-closure contains W_C also satisfies (2). In particular, $W_P = \{w \in W_C\colon w \text{ is a polynomial in } s \text{ variables}\}$ and, when $K = [0,1]$, $W_S = \{w \in W_\infty\colon w \text{ is a step function}\}$ satisfy (2). Hence, we answer the question above in the affirmative. Our proof involves an application of the Lyapunov theorem on vector measures which not only yields a more far reaching result than those of Kroó and Pinkus but also simplifies their proofs substantially.

2. Proof of Theorem 1. We shall make use of a lemma on moments which is somewhat more general than a similar lemma used by Kroó [**4**].

LEMMA. *Let (Ω, Σ, ν) be a finite, positive measure space, let*

$$S = \operatorname{span}\{s_1, \dots, s_n\}$$

be an n-dimensional subspace of $L^\infty(\Omega)$, let W be a convex cone in $L^\infty(\Omega)$ satisfying Condition (2) (with $K = \Omega$), and let

$$A_n = \left\{ \left(\int_\Omega w s_i \, d\nu \right)_{i=1}^n : w \in W \right\} \subseteq \mathbf{R}^n.$$

If S contains no nontrivial functions that are nonnegative ν-a.e. on Ω, then $A_n = \mathbf{R}^n$.

PROOF. Since W is a convex cone and is nonempty by (2), A_n is a nonempty convex cone in $\mathbf{R}^n$. Suppose $A_n \neq \mathbf{R}^n$. Then A_n has boundary point, say x. By the supporting hyperplane theorem, there exists a nontrivial linear functional φ on $\mathbf{R}^n$, given by $\varphi(\xi_i)_{i=1}^n = \sum_{i=1}^n \alpha_i \xi_i$, such that $\varphi(x) = \inf \varphi(A_n)$. Since A_n is a cone and φ is bounded below on A_n, $\inf \varphi(A_n) = 0$, and thus

$$0 \leq \varphi \left(\int_\Omega w s_i \, d\nu \right)_{i=1}^n = \int_\Omega w \left(\sum_{i=1}^n \alpha_i s_i \right) d\nu$$

for all $w \in W$. By (2), $s = \sum_{i=1}^n \alpha_i s_i \geq 0$ ν-a.e. on Ω which is a contradiction, and the lemma is proven.

We now define the A-property. For $f \in C(K)$, let $Z(f) = \{x \in K\colon f(x) = 0\}$ and $\operatorname{supp}(f) = K \backslash Z(f)$. For a subspace U of $C(K)$, let

$$U^* = \{u^* \in C(K)\colon |u^*| \equiv |u| \text{ on } K \text{ for some } u \in U\}.$$

DEFINITION. We say that a finite-dimensional subspace U of $C(K)$ satisfies the *A-property* (or is an *A-space*) if for every $u^* \in U^* \backslash \{0\}$ there exists $u \in U \backslash \{0\}$ such that $u = 0$ a.e. on $Z(u^*)$ and $uu^* \geq 0$ on K.

We shall use a standard characterization of best L^1-approximations and a characterization of the Chebyshev subspaces of $C_w(K)$ for fixed $w \in W_\infty$ due to Strauss [**9**]. Actually, Strauss proved Theorem 3 for $w = 1$ and $K = [0,1]$, but his proof readily yields the more general version.

THEOREM 2. *Let U be a subspace of $C(K)$, $w \in W_\infty$, and $f \in C(K) \backslash U$. Then 0 is a best approximation to f from U with respect to the norm $\|\cdot\|_w$ if and only if* *there exists* $\psi \in L^\infty(Z(f))$ *with* $|\psi| \le 1$ *such that*

$$\int_{\mathrm{supp}(f)} wu \,\mathrm{sgn} f \, d\mu + \int_{Z(f)} wu\psi \, d\mu = 0$$

for all $u \in U$.

THEOREM 3. *A finite-dimensional subspace* U *of* $C(K)$ *is Chebyshev in* $C_w(K)$, $w \in W_\infty$, *if and only if for every* $u^* \in U^*\backslash\{0\}$, 0 *is not a best approximation to* u^* *from* U *relative to the norm* $\|\cdot\|_w$.

PROOF OF THEOREM 1. Suppose U is Chebyshev in $C_w(K)$ for every $w \in W$, and let $u^* \in U^*\backslash\{0\}$. We have that $\sigma = \mathrm{sgn}\, u^*$ is continuous at each point of $\mathrm{supp}(u^*)$. Let $U_1 = \{u \in U : u = 0 \text{ a.e. on } Z(u^*)\}$. We need to show that there exists $u_1 \in U_1\backslash\{0\}$ such that $\sigma u_1 \ge 0$ on $\mathrm{supp}(u^*)$. Assume that no such u_1 exists. Let $\{g_1, \dots, g_k\}$ be a basis for U_1, and choose $g_{k+1}, \dots, g_n \in U$ so that $\{g_1, \dots, g_n\}$ is a basis for U. Letting $U_2 = \mathrm{span}\{g_{k+1}, \dots, g_n\}$, we have that $U = U_1 \oplus U_2$. By definition of U_1, if $u_2 \in U_2$ and $u_2 = 0$ a.e. on $Z(u^*)$, then $u_2 = 0$. We apply the Lyapunov theorem on vector measures to $g_{k+1}, \dots, g_n$ on $Z(u^*)$ to obtain a measurable function $\psi : Z(u^*) \to \{-1, 1\}$ such that

$$\int_{Z(u^*)} u_2 \psi \, d\mu = 0 \tag{3}$$

for all $u_2 \in U_2$. (See Lemma 2 in [**5**] for the precise version of Lyapunov's theorem used here.) For simplicity, we redefine $\sigma = \psi$ on $Z(u^*)$. By (3), if $u_2 \in U_2$ and $\sigma u_2 \ge 0$ a.e. on $Z(u^*)$, then $u_2 = 0$.

We now have that if $u \in U$ and $\sigma u \ge 0$ a.e. on K, then $u = 0$. To see this, write $u = u_1 + u_2$ whence $u_1 \in U_1$ and $u_2 \in U_2$ and suppose that $\sigma u \ge 0$ a.e. on K. Then $\sigma u_2 \ge 0$ a.e. on $Z(u^*)$ and thus $u_2 = 0$. Thus $\sigma u_1 \ge 0$ a.e. on K. Since σ is continuous at each point of $\mathrm{supp}(u^*)$, $\sigma u_1 \ge 0$ on $\mathrm{supp}(u^*)$, and by assumption, $u_1 = 0$.

We have that the finite-dimensional subspace $S = \{\sigma u : u \in U\}$ contains no nontrivial elements that are nonnegative a.e. on K. Moreover, each element of S is bounded, and by (2) and the Lemma there exists $w \in W$ such that

$$\int_K wu\sigma \, d\mu = 0 \tag{4}$$

for all $u \in U$. Since $\sigma = \mathrm{sgn}\, u^*$ on $\mathrm{supp}(u^*)$, (4) and Theorem 2 imply that 0 is a best approximation to u^* from U relative to $\|\cdot\|_w$, and by Theorem 3, U is not Chebyshev in $C_w(K)$, a contradiction. The proof of Theorem 1 is complete.

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