A Theorem on Weighted $L^1$-Approximation

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Abstract. It is proven that the A-property is necessary for a finite-dimensional subspace to be Chebyshev in $C(K)$ with respect to the weighted $L^1$-norm $\|f\|_w = \int_K w|f| d\mu$ for all weight functions $w$ in certain classes of functions.

1. Introduction. Let $K$ be a compact subset of $\mathbb{R}^s$ ($s \geq 1$) satisfying $K = \text{Int}(K)$ and denote by $C(K)$ the space of all continuous, real-valued functions on $K$. Let $W_\infty = \{w \in L^\infty(K) : w > 0 \text{ on } K\}$, and for $w \in W_\infty$ let $C_w(K)$ denote the space $C(K)$ with the $w$-weighted $L^1$-norm

$$
\|f\|_w = \int_K w|f| d\mu,
$$

where $\mu$ denotes Lebesgue measure. We say that a finite-dimensional subspace $U$ of $C(K)$ is Chebyshev in $C_w(K)$ if every $f \in C(K)$ has a unique best approximation from $U$ with respect to the norm (1).

Recent interest in the Chebyshev subspaces of $C(K)$ with $L^1$-norms was inspired by the discoveries that spaces of spline functions are Chebyshev in $C[0,1]$ in addition to the subspaces of $C[0,1]$ that satisfy the Haar condition on $(0,1)$ (see [8] and its references). A unifying feature of these spaces is the so-called A-property (defined in §2), and Strauss [10] proved that if $U$ satisfies the A-property, then $U$ is Chebyshev in $C[0,1]$. This result is easily generalized for any $w \in W_\infty$ and any $K$ as above. When $K = [0,1]$, Kroó [1] established a converse showing that if $U$ is Chebyshev in $C_w[0,1]$ for all $w \in W_B = \{w \in W_\infty : \inf w > 0\}$, then $U$ is an A-space, and Sommer [7] generalized this result to the multivariate setting. Independent of Kroó, Pinkus [6] sought a converse using only the continuous weight functions and succeeded under the additional assumption that $\mu(Z(u)) = \mu(\text{Int}(Z(u)))$ for all $u \in U$ where $Z(u) = \{x \in K : u(x) = 0\}$. Subsequently, Kroó [3] removed this condition for any $K$ as above showing that the A-property is necessary for $U$ to be Chebyshev in $C_w(K)$ for all $w \in W_C = \{w \in C(K) : w > 0 \text{ on } K\}$. It is natural to ask whether the analytic or even the polynomial weight functions suffice. Indeed, when $K = [0,1]$ and $U$ satisfied Pinkus' condition, Kroó [2] showed that the $C^\infty$-weight functions suffice. In this note we give general conditions on $W \subseteq W_\infty$ so that the A-property is necessary for $U$ to be Chebyshev in $C_w(K)$ for all $w \in W$.

**Theorem 1.** Let $W$ be a conex cone in $W_\infty$ satisfying the condition

$$
\text{if } q \text{ is a bounded, measurable function and } \int_K wq \, d\mu \geq 0 \text{ for all } w \in W, \text{ then } q \geq 0 \text{ a.e. on } K.
$$

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If a finite-dimensional subspace $U$ of $C(K)$ is Chebyshev in $C_w(K)$ for all $w \in W$, then $U$ is an $A$-space.

Evidently $W_\infty$ satisfies condition (2) and an argument using Lusin’s theorem shows that $W_C$ also satisfies (2). Moreover any convex cone $W$ in $W_\infty$ whose $L^\infty$-closure contains $W_C$ also satisfies (2). In particular, $W_P = \{ w \in W_C : w$ is a polynomial in $s$ variables $\}$ and, when $K = [0,1], W_S = \{ w \in W_\infty : w$ is a step function $\}$ satisfy (2). Hence, we answer the question above in the affirmative. Our proof involves an application of the Lyapunov theorem on vector measures which not only yields a more far reaching result than those of Kroó and Pinkus but also simplifies their proofs substantially.

2. Proof of Theorem 1. We shall make use of a lemma on moments which is somewhat more general than a similar lemma used by Kroó [4].

**Lemma.** Let $(\Omega, \Sigma, \nu)$ be a finite, positive measure space, let

$$S = \text{span}\{s_1, \ldots, s_n\}$$

be an $n$-dimensional subspace of $L^\infty(\Omega)$, let $W$ be a convex cone in $L^\infty(\Omega)$ satisfying Condition (2) (with $K = \Omega$), and let

$$A_n = \left\{ \left( \int_\Omega w_s dv \right)^n : w \in W \right\} \subseteq \mathbb{R}^n.$$

If $S$ contains no nontrivial functions that are nonnegative $\nu$-a.e. on $\Omega$, then $A_n = \mathbb{R}^n$.

**Proof.** Since $W$ is a convex cone and is nonempty by (2), $A_n$ is a nonempty convex cone in $\mathbb{R}^n$. Suppose $A_n \neq \mathbb{R}^n$. Then $A_n$ has boundary point, say $x$. By the supporting hyperplane theorem, there exists a nontrivial linear functional $\varphi$ on $\mathbb{R}^n$, given by $\varphi(\xi_i)_{i=1}^n = \sum_{i=1}^n \alpha_i \xi_i$, such that $\varphi(x) = \inf \varphi(A_n)$. Since $A_n$ is a cone and $\varphi$ is bounded below on $A_n$, $\inf \varphi(A_n) = 0$, and thus

$$0 \leq \varphi \left( \int_\Omega w_s dv \right)^n_{i=1} = \int_\Omega w \left( \sum_{i=1}^n \alpha_i s_i \right) dv$$

for all $w \in W$. By (2), $s = \sum_{i=1}^n \alpha_i s_i \geq 0 \nu$-a.e. on $\Omega$ which is a contradiction, and the lemma is proven.

We now define the $A$-property. For $f \in C(K)$, let $Z(f) = \{ x \in K : f(x) = 0 \}$ and $\text{supp}(f) = K \setminus Z(f)$. For a subspace $U$ of $C(K)$, let

$$U^* = \{ u^* \in C(K) : |u^*| \equiv |u| \text{ on } K \text{ for some } u \in U \}.$$

**Definition.** We say that a finite-dimensional subspace $U$ of $C(K)$ satisfies the $A$-property (or is an $A$-space) if for every $u^* \in U^* \setminus \{0\}$ there exists $u \in U \setminus \{0\}$ such that $u = 0$ a.e. on $Z(u^*)$ and $uu^* \geq 0$ on $K$.

We shall use a standard characterization of best $L^1$-approximations and a characterization of the Chebyshev subspaces of $C_w(K)$ for fixed $w \in W_\infty$ due to Strauss [9]. Actually, Strauss proved Theorem 3 for $w = 1$ and $K = [0,1]$, but his proof readily yields the more general version.

**Theorem 2.** Let $U$ be a subspace of $C(K), w \in W_\infty$, and $f \in C(K) \setminus U$. Then $0$ is a best approximation to $f$ from $U$ with respect to the norm $\| \cdot \|_w$ if and only if
there exists $\psi \in L^\infty(Z(f))$ with $|\psi| \leq 1$ such that

$$\int_{\text{supp}(f)} wu \text{sgn} f \, d\mu + \int_{Z(f)} wu \psi \, d\mu = 0$$

for all $u \in U$.

**Theorem 3.** A finite-dimensional subspace $U$ of $C(K)$ is Chebyshev in $C_w(K)$, $w \in W_\infty$, if and only if for every $u^* \in U^\ast \setminus \{0\}$, 0 is not a best approximation to $u^*$ from $U$ relative to the norm $\| \cdot \|_w$.

**Proof of Theorem 1.** Suppose $U$ is Chebyshev in $C_w(K)$ for every $w \in W$, and let $u^* \in U^\ast \setminus \{0\}$. We have that $\sigma = \text{sgn} u^*$ is continuous at each point of $\text{supp}(u^*)$. Let $U_1 = \{u \in U : u = 0 \text{ a.e. on } Z(u^*)\}$. We need to show that there exists $u_1 \in U_1 \setminus \{0\}$ such that $\sigma u_1 \geq 0$ on $\text{supp}(u^*)$. Assume that no such $u_1$ exists. Let $\{g_1, \ldots, g_k\}$ be a basis for $U_1$, and choose $g_{k+1}, \ldots, g_n \in U$ so that $\{g_1, \ldots, g_n\}$ is a basis for $U$. Letting $U_2 = \text{span}\{g_{k+1}, \ldots, g_n\}$, we have that $U = U_1 \oplus U_2$. By definition of $U_1$, if $u_2 \in U_2$ and $u_2 = 0$ a.e. on $Z(u^*)$, then $u_2 = 0$. We apply the Lyapunov theorem on vector measures to $g_{k+1}, \ldots, g_n$ on $Z(u^*)$ to obtain a measurable function $\psi : Z(u^*) \to \{-1, 1\}$ such that

$$\int_{Z(u^*)} u_2 \psi \, d\mu = 0$$

for all $u_2 \in U_2$. (See Lemma 2 in [5] for the precise version of Lyapunov's theorem used here.) For simplicity, we redefine $\sigma = \psi$ on $Z(u^*)$. By (3), if $u_2 \in U_2$ and $\sigma u_2 \geq 0$ a.e. on $Z(u^*)$, then $u_2 = 0$.

We now have that if $u \in U$ and $\sigma u \geq 0$ a.e. on $K$, then $u = 0$. To see this, write $u = u_1 + u_2$ whence $u_1 \in U_1$ and $u_2 \in U_2$ and suppose that $\sigma u \geq 0$ a.e. on $K$. Then $\sigma u_2 \geq 0$ a.e. on $Z(u^*)$ and thus $u_2 = 0$. Thus $\sigma u_1 \geq 0$ a.e. on $K$. Since $\sigma$ is continuous at each point of $\text{supp}(u^*)$, $\sigma u_1 \geq 0$ on $\text{supp}(u^*)$, and by assumption, $u_1 = 0$.

We have that the finite-dimensional subspace $S = \{\sigma u : u \in U\}$ contains no nontrivial elements that are nonnegative a.e. on $K$. Moreover, each element of $S$ is bounded, and by (2) and the Lemma there exists $w \in W$ such that

$$\int_K wu \sigma \, d\mu = 0$$

for all $u \in U$. Since $\sigma = \text{sgn} u^*$ on $\text{supp}(u^*)$, (4) and Theorem 2 imply that 0 is a best approximation to $u^*$ from $U$ relative to $\| \cdot \|_w$, and by Theorem 3, $U$ is not Chebyshev in $C_w(K)$, a contradiction. The proof of Theorem 1 is complete.

**References**


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