PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 1, September 1987

ASYMPTOTIC INTEGRATION OF A SECOND ORDER ORDINARY DIFFERENTIAL EQUATION

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ABSTRACT. Equation (1) $(r(t)x')' + f(t)x = 0$ is regarded as a perturbation of (2) $(r(t)y')' + g(t)y = 0$, where the latter is nonoscillatory at infinity. It is shown that if a certain improper integral involving $f - g$ converges sufficiently rapidly (but perhaps conditionally), then (1) has a solution which behaves for large t like a principal solution of (2). The proof of this result is presented in such a way that it also yields as a by-product an improvement on a recent related result of Trench.

Let f, g, and r be continuous on $[a, \infty)$. Consider the scalar differential equation

$$(r(t)x')' + f(t)x = 0 \qquad (r(t) > 0, t \geq a) \tag{1}$$

as a perturbation of

$$(r(t)y')' + g(t)y = 0. \tag{2}$$

The quantities x and f in (1) may be complex-valued, while g and y in (2) are necessarily real-valued. Assume also that (2) is nonoscillatory at infinity. Then (2) has solutions y_1 and y_2 which are positive on $[b, \infty)$ for some $b \geq a$ and satisfy

$$r(y_1 y_2' - y_1' y_2) = 1, \tag{3}$$

and the function

$$\rho = y_2/y_1 \tag{4}$$

approaches infinity monotonically as $t \to \infty$. The solution y_1 is said to be a *principal* solution of (2) (see [**1**, p. 355]).

The following problem has already received much attention: What conditions on $f - g$ imply that (1) has a solution x_1 which behaves for large t like y_1? It seems to be *reasonable* to assume that the improper integrals

$$G_1(t) = \int_t^\infty y_1 y_2 (f - g)\, ds \quad \text{and} \quad G_2(t) = \int_t^\infty y_1^2 (f - g)\, ds \tag{5}$$

converge (perhaps conditionally). As noticed in [**2**], G_2 exists and satisfies

$$|G_2(t)| \leq \frac{2}{\rho(t)} \sup_{s \geq t} |G_1(s)| \qquad (t \geq b), \tag{6}$$

provided that G_1 exists.

The following is our main result. Throughout the paper, φ is positive and nonincreasing on $[a, \infty)$, with $\varphi(t) \to 0$ as $t \to \infty$. The symbol "O" refers to behavior as $t \to \infty$.

Received by the editors May 27, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 34C99.

THEOREM 1. *Suppose that* G_1 *exists and satisfies*

$$G_1 = O(\varphi). \tag{7}$$

If

$$\int^{\infty} |G_1| \frac{\rho'}{\rho} \varphi \, dt < \infty, \tag{8}$$

then (1) *has a solution* x_1 *satisfying*

$$x_1 = [1 + O(\varphi_1)] y_1 \tag{9}$$

and

$$(x_1/y_1)' = O(\varphi \rho'/\rho) \tag{10}$$

where

$$\varphi_1(t) = \max \left\{ \varphi(t), \int_t^{\infty} |G_1| \frac{\rho'}{\rho} \varphi \, ds \right\}. \tag{11}$$

We take a more general approach in the proof to obtain a useful by-product (Theorem 2) which improves upon a recent result of Trench [**2**]. We present also an example showing that Theorem 1 is applicable where Theorem 2 is not.

THEOREM 2. *Suppose that* G_1 *exists and satisfies* (7). *If*

$$\int^{\infty} |G_2| \rho' \varphi \, ds < \infty \tag{12}$$

and

$$\overline{\lim_{t \to \infty}} (\varphi(t))^{-1} \int_t^{\infty} |G_2| \rho' \varphi \, ds < 1, \tag{13}$$

then (1) *has a solution* x_1 *satisfying* (10) *and*

$$x_1 = [1 + O(\varphi)] y_1. \tag{14}$$

Actually, the result of [**2**] is that Theorem 2 holds if the left side of (13) is smaller than 1/3.

REMARK. By an argument given in [**2**], it is possible to choose constants c and b_1 $(\geq b)$ so that the function

$$x_2(t) = x_1(t) \left[c + \int_{b_1}^{t} \frac{ds}{r x_1^2} \right]$$

is a solution of (1) and behaves like y_2 as $t \to \infty$. We omit this discussion here.

We will combine the proofs of Theorems 1 and 2. It is convenient to define

$$(\mathcal{M}_1 z)(t) = \int_t^{\infty} y_1 y_2 (f - g) z \, ds \qquad (t \geq b) \tag{15}$$

and

$$(\mathcal{M}_2 z)(t) = \int_t^{\infty} y_1^2 (f - g) z \, ds \qquad (t \geq b). \tag{16}$$

Following Trench [**2**], the solution x_1 of (1) will be found in the form

$$x_1 = (1 + z_1) y_1, \tag{17}$$

where z_1 is the fixed point (function) of the mapping

$$\mathcal{T} z = G_1 - \rho G_2 + \mathcal{M}_1 z - \rho \mathcal{M}_2 z. \tag{18}$$

We let $\mathcal{T}$ act on the Banach space

$$B = \{z \in C'[t_0, \infty) | z = O(\varphi_1),\ z' = O(\varphi\rho'/\rho)\} \tag{19}$$

with norm

$$\|z\| = \sup_{t \geq t_0} \max \left\{ \frac{|z|}{\varphi_1}, \frac{|z'|\rho}{K\varphi\rho'} \right\}, \tag{20}$$

where the function φ_1 and the constants t_0 and K will be specified later. (Trench [**2**] considered the case $\varphi_1 = \varphi$ and $K = 2$.) Now we suppose only that $t \geq t_0 \geq b$, $K > 0$, $\varphi_1(t) \geq \varphi(t)$, and $\varphi_1 \to 0$ monotonically as $t \to \infty$. Finally, suppose that G_j in (5) exist and satisfy

$$|G_1(t)| + \rho(t)|G_2(t)| \leq \varphi(t) \tag{21}$$

(see (6) and (7)), otherwise φ is replaced by $c\varphi$, where c is a sufficiently large constant. Then $\mathcal{M}_2 z$ exists and satisifies

$$|(\mathcal{M}_2 z)(t)| \leq \|z\| [\varphi_1(t) + K\varphi(t)] \varphi(t)/\rho(t) \tag{22}$$

for any $z \in B$. To see this, we first observe from (20) that

$$|z| \leq \|z\| \varphi_1 \quad \text{and} \quad |z'| \leq K\|z\|\varphi\rho'/\rho. \tag{23}$$

Rewriting (16) and integrating by parts yields

$$(\mathcal{M}_2 z)(t) = -\int_t^\infty G_2' z\, ds = G_2(t) z(t) + \int_t^\infty G_2 z'\, ds.$$

In view of (21) and (23), this integration is valid, since $|G_2 z| \leq \|z\|\varphi_1\varphi/\rho$, the integral on the right side converges absolutely, and is dominated by

$$K\|z\|\varphi^2(t) \int_t^\infty \frac{\rho'}{\rho^2} ds = \frac{K\|z\|\varphi^2(t)}{p(t)}.$$

This implies (22). Similarly, using the equalities

$$(\mathcal{M}_1 z)(t) = G_2(t)\rho(t)z(t) + \int_t^\infty G_2(\rho z)'\, ds$$

and

$$(\mathcal{M}_1 z)(t) = G_1(t) z(t) + \int_t^\infty G_1 z'\, ds$$

(see (4), (5), and (15)), it is easy to check that (21) and (23) imply the existence of $\mathcal{M}_1 z$, with

$$|(\mathcal{M}_1 z)(t)| \leq \|z\| \left[\varphi_1(t)\varphi(t) + \int_t^\infty |G_2|\rho'(\varphi_1 + K\varphi)\, ds \right] \tag{24}$$

or

$$|(\mathcal{M}_1 z)(t)| \leq \|z\| \left[\varphi_1(t)\varphi(t) + K \int_t^\infty |G_1| \frac{\rho'}{\rho} \varphi\, ds \right] \tag{25}$$

for any $z \in B$, provided that the corresponding right side is finite.

Differentiating in (5), (15), and (16) yields

$$(26)\qquad (G_1 - \rho G_2)' = -\rho' G_2 \quad \text{and} \quad (\mathcal{M}_1 z - \rho \mathcal{M}_2 z)' = -\rho' \mathcal{M}_2 z.$$

To prove Theorem 1, assume that (8) holds, and choose φ_1 as in (11). Then the right side of (25) is dominated by $\|z\|(K+\varphi)\varphi_1$. Consequently, (22), (25), and (26) imply that the function

$$(27)\qquad \mathcal{L}z = \mathcal{M}_1 z - \rho \mathcal{M}_2 z$$

is continuously differentiable and satisfies

$$(28)\qquad \sup_{t \geq t_0} \frac{|\mathcal{L}z|}{\varphi_1} \leq \|z\|[2\varphi(t_0) + K + K\varphi(t_0)]$$

and

$$(29)\qquad \sup_{t \geq t_0} \frac{|(\mathcal{L}z)'|\rho}{K\varphi\rho'} \leq \|z\|[\varphi(t_0) + K^{-1}\varphi_1(t_0)].$$

Now we can choose $K = 1/2$ and a large t_0 so that the right sides of (28) and (29) are dominated by $A\|z\|$, with a constant $A < 1$. This and definition (20) imply that $\|\mathcal{L}z\| \leq A\|z\|$ for any $z \in B$. From (19), (21), and (26) we see that $(G_1 - \rho G_2) \in B$. Thus $\mathcal{T}$ in (18) is a contraction on B. If z_1 is the (unique) fixed point of $\mathcal{T}$, then x_1 in (17) is a solution of (1) satisfying (9) and (10), because of $z_1 \in B$. This completes the proof Theorem 1.

To prove Theorem 2, we put $\varphi_1 = \varphi$. If (12) holds, then (22), (24), and (26) imply that the function $\mathcal{L}z$ in (27) is continuously differentiable and satisfies

$$(30)\qquad \sup_{t \geq t_0} \frac{|\mathcal{L}z|}{\varphi} \leq \|z\| \left[(K+1) \sup_{t \geq t_0} (\varphi(t))^{-1} \int_t^\infty |G_2|\rho'\varphi\, ds + (K+2)\varphi(t_0) \right]$$

and

$$(31)\qquad \sup_{t \geq t_0} \frac{|(\mathcal{L}z)'|\rho}{K\varphi\rho'} \leq \|z\|(1 + K^{-1})\varphi(t_0).$$

Assuming (13) holds, we now choose $K > 0$ so that

$$(K+1) \varlimsup_{t \to \infty} (\varphi(t))^{-1} \int_t^\infty |G_2|\rho'\varphi\, ds < 1.$$

Since $\varphi(t_0) \to 0$ as $t_0 \to \infty$, there exists a t_0 so that the right sides of (30) and (31) are dominated by $A\|z\|$, with a constant $A < 1$. The rest of the proof is the same as that of Theorem 1.

EXAMPLE. Trench [**2**] has considered the equation

$$(32)\qquad x'' + K[t^{-1}(\log t)^{-\alpha} \sin t]x = 0,$$

where K and α are nonzero constants, as a perturbation of $y'' = 0$. Then $y_1 = 1$, $y_2 = \rho = t$, and

$$\begin{aligned} G_1(t) &= K \int_t^\infty (\log s)^{-\alpha} \sin s\, ds \\ &= [K \cos t + O(t^{-1})](\log t)^{-\alpha}, \end{aligned}$$

provided that $\alpha > 0$ (see (5) and use integration by parts). Thus (7) holds with $\varphi = (\log t)^{-\alpha}$. For this φ and $\alpha > 1/2$,

$$\int_t^\infty \frac{\rho'}{\rho}\varphi^2\,ds = \int_t^\infty \frac{(\log s)^{-2\alpha}}{s}ds = \frac{(\log t)^{1-2\alpha}}{2\alpha - 1},$$

which proves (8) because (7). Theorem 1 implies that (32) has a solution x_1 satisfying

$$x_1(t) = 1 + O((\log t)^{-\beta}) \quad \text{and} \quad x_1'(t) = O(t^{-1}(\log t)^{-\alpha})$$

with $\beta = \min\{\alpha, 2\alpha - 1\}$, provided that $\alpha > 1/2$. As shown in [**2**], the left side of (13) is not finite here unless $\alpha \geq 1$. So Theorem 2 does not apply if $\alpha < 1$.

ACKNOWLEDGMENT. I wish to thank to Professor Trench for a preprint of [**2**] and for many helpful suggestions on the present work.

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