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DENTING POINTS IN TENSOR PRODUCTS OF BANACH SPACES

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ABSTRACT. Let $\operatorname{dent} A$ denote the set of denting points of a subset A of some Banach space. We prove

$$\operatorname{dent} \operatorname{cl} \operatorname{co}(K \otimes L) = \operatorname{dent} K \otimes \operatorname{dent} L$$

for closed, bounded, absolutely convex subsets K and L of Banach spaces X and Y. Here the closure refers to the completion of $X \otimes Y$ w.r.t. some reasonable crossnorm.

1. Introduction. In this note we consider denting points in subsets of tensor products of Banach spaces. Let us recall that x is a denting point of a subset K of a Banach space X ($x \in \operatorname{dent} K$ for short) if $x \notin \operatorname{cl} \operatorname{co}(K \backslash B_\varepsilon(x))$ for all $\varepsilon > 0$. Here $B_\varepsilon(x)$ denotes the closed ball with radius ε centered at x, we write B_X for the closed unit ball of X. The reader who is interested in a discussion of the relevance of denting points in connection with the Radon-Nikodým property (RNP) is referred to the monographs [**2** and **3**].

Further, recall that a norm α on the algebraic tensor product $X \otimes Y$ of two Banach spaces is called reasonable if $\|x \otimes y\| = \|x\| \cdot \|y\|$ for all $x \in X$, $y \in Y$, and if $x^* \otimes y^*$, considered as a functional on the α-normed tensor product of X and Y, is continuous with norm $\|x^*\| \cdot \|y^*\|$ for all $x^* \in X^*$, $y^* \in Y^*$. The completed α-normed tensor product is denoted by $X \hat{\otimes}_\alpha Y$. ([**3**, Chapter VIII] is a good source for the prerequisites on tensor products which are needed.)

The main result is as follows.

THEOREM 1. *Let K and L be closed, bounded, absolutely convex subsets of Banach spaces X and Y, and let α be a reasonable crossnorm. Then*

$$\operatorname{dent} \operatorname{cl} \operatorname{co}(K \otimes L) = \operatorname{dent} K \otimes \operatorname{dent} L,$$

where the closure refers to $X \hat{\otimes}_\alpha Y$ and $K \otimes L = \{x \otimes y \colon x \in K,\ y \in L\}$.

2. Proof of Theorem 1. We need two preparatory lemmas for the proof. In the following discussion let K be a closed, bounded, absolutely convex subset of some real Banach space X, and let $x_0 \in \operatorname{dent} K$. For $\varepsilon > 0$ define $K_\varepsilon := \operatorname{cl} \operatorname{co}(K \backslash B_\varepsilon(x_0))$. Using the Hahn-Banach theorem it is possible to find a functional $x^* \in X^*$ and a number $\delta > 0$ (both dependent on ε in general) such that $x^*(x_0) = 1$ and $x^*(x) \leq 1 - \delta$ for all $x \in K_\varepsilon$. It will be convenient to use the notations $K_0 := K$ and Φ_ε ($\varepsilon \geq 0$) for the real-valued norm-continuous function defined by $\Phi_\varepsilon(x^*) := \sup\{x^*(x) \colon x \in K_\varepsilon\}$. The next lemma shows that it is possible to choose a separating functional as above with $\Phi_0(x^*)$ small.

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LEMMA 2. *For $x_0 \in \operatorname{dent} K$ and $\varepsilon > 0$ there exist $\delta > 0$ and $x^* \in X^*$ such that*
(i) $x^*(x_0) = 1$,
(ii) $\Phi_\varepsilon(x^*) = 1 - \delta$,
(iii) $\Phi_0(x^*) \leq 1 + \varepsilon\delta$.

PROOF. Choose a positive number α so small that $\alpha/(1-\alpha) < \varepsilon$, and choose β to satisfy $0 < \beta < \varepsilon\alpha/2$. Separate x_0 strictly from K_β by means of a functional x^*. Of course, we may assume $x^*(x_0) = 1$ and $x^*(x) < 1$ for $x \in K_\beta$, since K is absolutely convex. Define δ by $\delta := 1 - \Phi_\varepsilon(x^*)$. As x^* was supposed to separate strictly, we have $\delta > 0$. It is left to estimate $\Phi_0(x^*)$, that is $x^*(x)$ for $x \in K$.

It is enough to do so for $x \in K \cap B_\beta(x_0)$, the remaining x's satisfy $x^*(x) \leq 1$ anyway. Now consider an arbitrary $y \in K$ with $\|y - x_0\| > \varepsilon$. Then

$$\|\alpha y + (1-\alpha)x - x_0\| > \beta.$$

In fact, otherwise we would have (taking into account $\|x - x_0\| \leq \beta$)

$$\begin{aligned}\|\alpha(y - x_0)\| &\leq \|\alpha y + (1-\alpha)x - x_0\| + \|(1-\alpha)(x - x_0)\| \\ &\leq \beta + (1-\alpha)\beta < \alpha\varepsilon,\end{aligned}$$

which contradicts the choice of y. Hence,

$$x^*(\alpha y + (1-\alpha)x) \leq 1.$$

Taking the supremum over all $y \in K \backslash B_\varepsilon(x_0)$, which amounts to the same as the supremum over all $y \in K_\varepsilon$, we obtain

$$\alpha\Phi_\varepsilon(x^*) + (1-\alpha)x^*(x) \leq 1$$

and thus $x^*(x) \leq 1 + \varepsilon\delta$ by the choice of δ and α.

We need a stronger version of Lemma 2. Note that we have actually proved the following statement: Let β be as in the above proof, and let

$$\mathcal{C} := \{x^* \in X^* \colon x^*(x_0) = 1, \Phi_\beta(x^*) \leq 1\}.$$

Then $\Phi_0(x^*) \leq 1 + \varepsilon\delta$ for $x^* \in \mathcal{C}$ and $\delta := 1 - \Phi_\varepsilon(x^*)$.

LEMMA 3. *For $x_0 \in \operatorname{dent} K$ and $\varepsilon > 0$ there exists $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$ there is $x^* \in X^*$ with*
(i) $x^*(x_0) = 1$.
(ii) *If $x^*(x) > 1 - \delta$ and $x \in K$, then $\|x - x_0\| \leq \varepsilon$.*
(iii) *$x^*(x) \leq 1 + \varepsilon\delta$ for all $x \in K$.*

PROOF. The second condition is nothing but $\Phi_\varepsilon(x^*) \leq 1 - \delta$. Let β and $\mathcal{C}$ be as above, and let $\eta := \inf\{1 - \Phi_\varepsilon(x^*) : x^* \in \mathcal{C}\}$.

First case: $\eta > 0$. Put $\delta_0 := \eta$. Given $0 < \delta \leq \delta_0$ consider $\varepsilon^* = \min(\beta, \varepsilon\delta)$. Lemma 2 (with ε^* instead of ε) yields $x^* \in X^*$ and $\delta^* > 0$ with $x^*(x_0) = 1$ and $\Phi_\beta(x^*) \leq \Phi_{\varepsilon^*}(x^*) \leq 1$, hence $x^* \in \mathcal{C}$. Therefore, $1 - \Phi_\varepsilon(x^*) \geq \eta \geq \delta$. x^* satisfies the third condition, too, since by Lemma 2

$$\Phi_0(x^*) \leq 1 + \varepsilon^*\delta^* \leq 1 + \varepsilon\delta.$$

Second case: $\eta = 0$. Choose $\delta_0 > 0$ and $x_0^* \in \mathcal{C}$ according to Lemma 2, in particular $\Phi_\varepsilon(x_0^*) = 1 - \delta_0$. The continuous funciton $\mathbf{1} - \Phi_\varepsilon$ maps the convex set $\mathcal{C}$ onto an interval, hence $]0, \delta_0] \subset (\mathbf{1} - \Phi_\varepsilon)(\mathcal{C})$. In other words, for $0 < \delta \leq \delta_0$ there exists $x^* \in \mathcal{C}$ with $\Phi_\varepsilon(x^*) = 1 - \delta$. According to our above remarks, x^* fulfills the conclusions of Lemma 3.

PROOF OF THEOREM 1. We restrict ourselves to the case of real Banach spaces, the complex case can be established along the same lines. Moreover, we assume that K and L are contained in the respective unit balls of X and Y.

Let $x_0 \in \operatorname{dent} K, y_0 \in \operatorname{dent} L$, and $\varepsilon > 0$. Using Lemma 2 we find $x^* \in X^*$, $y^* \in Y^*$, $\delta_1 > 0$, $\delta_2 > 0$ with the properties

$$\begin{aligned}
&x^*(x_0) = y^*(y_0) = 1,\\
&x \in K \text{ and } x^*(x) > 1 - \delta_1 \text{ imply } \|x - x_0\| \le \varepsilon,\\
&y \in L \text{ and } y^*(y) > 1 - \delta_2 \text{ imply } \|y - y_0\| \le \varepsilon,\\
&x^*(x) \le 1 + \varepsilon\delta_1 \text{ for all } x \in K,\\
&y^*(y) \le 1 + \varepsilon\delta_2 \text{ for all } y \in L.
\end{aligned}$$

Lemma 3 permits us to assume $\delta_1 = \delta_2 =: \delta$.

In the first step we are going to prove

$$\begin{gathered}
x \in K,\ y \in L, \text{ and } \langle x^* \otimes y^*,\ x \otimes y\rangle > 1 - \delta/2 \text{ imply}\\
\|x \otimes y - x_0 \otimes y_0\|_\alpha \le 2\varepsilon.
\end{gathered} \tag{1}$$

In fact,

$$\begin{aligned}
\delta/2 &> \langle x^* \otimes y^*, x_0 \otimes y_0 - x \otimes y\rangle\\
&= (1 - x^*(x))y^*(y) + (1 - y^*(y))\\
&\ge (-\varepsilon\delta)y^*(y) + (1 - y^*(y)) \quad (\text{w.l.o.g. } y^*(y) \ge 0)\\
&\ge -\varepsilon\delta(1 + \varepsilon\delta) + (1 - y^*(y)),
\end{aligned}$$

hence

$$1 - y^*(y) < \delta/2 + \varepsilon\delta(1 + \varepsilon\delta) < \delta$$

for ε small enough. It follows that $\|y - y_0\| \le \varepsilon$. Analogously we deduce $\|x - x_0\| \le \varepsilon$, and we end up with

$$\|x \otimes y - x_0 \otimes y_0\|_\alpha \le \|x \otimes (y - y_0)\|_\alpha + \|(x - x_0) \otimes y_0\|_\alpha \le 2\varepsilon.$$

In the second step we shall prove

$$\begin{gathered}
u \in \operatorname{co}(K \otimes L) \text{ and } \langle x^* \otimes y^*, u\rangle > 1 - \varepsilon\delta/2 \text{ imply}\\
\|u - x_0 \otimes y_0\|_\alpha \le 16\varepsilon,
\end{gathered} \tag{2}$$

which is enough to show $x_0 \otimes y_0 \in \operatorname{dent} \operatorname{cl} \operatorname{co}(K \otimes L)$. To prove (2), consider a convex combination $u = \sum_N \lambda_i \cdot x_i \otimes y_i \in \operatorname{co}(K \otimes L)$, and define the following subsets of $N = \{1, \dots, n\}$:

$$\begin{aligned}
I &= \{i \in N \colon \|x_i \otimes y_i - x_0 \otimes y_0\|_\alpha \le 2\varepsilon\},\\
J &= \{i \in N \colon \|x_i \otimes y_i - x_0 \otimes y_0\|_\alpha > 2\varepsilon\},\\
J' &= \{i \in N \colon \langle x^* \otimes y^*, x_i \otimes y_i\rangle \le 1 - \delta/2\},\\
R &= \{i \in N \colon \langle x^* \otimes y^*, x_i \otimes y_i\rangle > 1\}.
\end{aligned}$$

$J \subset J'$ by (1), therefore

$$\begin{aligned}\sum_J \lambda_i &\le \sum_{J'} \lambda_i \\ &\le \frac{2}{\delta} \sum_{J'} \lambda_i \cdot (1 - \langle x^* \otimes y^*, x_i \otimes y_i \rangle) \\ &\le \frac{2}{\delta} \left(\sum_N - \sum_R \right) (\lambda_i \cdot (1 - \langle x^* \otimes y^*, x_i \otimes y_i \rangle)) \\ &\le \frac{2}{\delta} \Bigg(1 - \langle x^* \otimes y^*, u \rangle \\ &\qquad + \sum_R \lambda_i \cdot (\sup\{x^*(x) \colon x \in K\} \cdot \sup\{y^*(y) \colon Y \in L\} - 1) \Bigg) \\ &\le \frac{2}{\delta} (\varepsilon\delta/2 + (1 + \varepsilon\delta)^2 - 1) \le 7\varepsilon.\end{aligned}$$

Consequently,

$$\|u - x_0 \otimes y_0\|_\alpha \le \left(\sum_I + \sum_J \right) (\lambda_i \|x_i \otimes y_i - x_0 \otimes y_0\|_\alpha) \le 2\varepsilon + 7\varepsilon \cdot 2 = 16\varepsilon.$$

Conversely, let $v \in \operatorname{dent} \operatorname{cl} \operatorname{co}(K \otimes L)$. Since $\operatorname{dent} \operatorname{cl} \operatorname{co} D \subset \operatorname{cl} D$ holds for any bounded subset D of any Banach space, and since $K \otimes L$ is easily seen to be closed in $X \hat{\otimes}_\alpha Y$, it follows that $v \in K \otimes L$. Thus, $v = x_0 \otimes y_0$ for some $x_0 \in K$, $y_0 \in L$. Necessarily x_0 and y_0 have to be denting points of K and L.

It should be pointed out that Theorem 1 does not extend to merely convex closed bounded sets K or L. In fact, if K and L are closed, bounded, and convex with K, in addition, absolutely convex, then $M := \operatorname{cl} \operatorname{co} K \otimes L$ is absolutely convex, too, so that (by Theorem 1) $x \otimes y \in \operatorname{dent} M$ if and only if $x \in \operatorname{dent} K$ and $y \in \operatorname{dent} \operatorname{aco} L$. Of course, the inclusion $\operatorname{dent} M \subset \operatorname{dent} K \otimes \operatorname{dent} L$ remains valid also in the general case.

Let us draw some easy conclusions from Theorem 1. Since the unit ball of the projective tensor product is by definition $\operatorname{cl} \operatorname{co}(B_X \otimes B_Y)$ we obtain

COROLLARY 4. $\operatorname{dent} B_{X \hat{\otimes}_\pi Y} = \operatorname{dent} B_X \otimes \operatorname{dent} B_Y$.

This result was proved in [**8**] in an entirely different manner. In connection with the discussion of RNP it is interesting to note that the projective tensor product of spaces with RNP need not have RNP [**1**]. Moreover, Theorem 1 and Corollary 4 remain true if "denting point" is replaced by "strongly exposed point" (in the latter case the separating functional may be chosed independently of ε). Thus, we recover the result of [**7**] which has proved useful in the study of Banach spaces, in particular spaces of operators, e.g. [**4, 5, 6**].

Our final remark concerns extreme functionals on spaces of bounded linear operators (cf. also [**8**] for interesting results). Under the identification of the Banach space V with a subsapce of V^{**} the inclusion $\operatorname{dent} B_V \subset \operatorname{dent} B_{V^{**}}$ is always valid (quickest proof by means of the principle of local reflexivity). In particular it follows from Corollary 4 that

$$\begin{aligned}\operatorname{dent} B_{X^{**}} \otimes \operatorname{dent} B_{Y^{**}} &\subset \operatorname{dent} B_{X^{**} \hat{\otimes}_\pi Y^{**}} \\ &\subset \operatorname{dent} B_{(X^{**} \hat{\otimes}_\pi Y^{**})^{**}} = \operatorname{dent} B_{L(X^{**}, Y^{***})^*}\end{aligned}$$

so that

$$\operatorname{dent} B_X \otimes \operatorname{dent} B_Y \subset \operatorname{dent} B_{X^{**}} \otimes \operatorname{dent} B_{Y^{**}} \subset \operatorname{dent} B_{L(X,Y^*)^*},$$

as $L(X, Y^*)^*$ is isometric to a subspace of $L(X^{**}, Y^{***})^*$ containing $X^{**} \otimes Y^{**}$. A similar result does not hold for extreme points. P. Harmand has found an example which shows that $T \to \langle T^{**}p, q\rangle$ need not be an extreme functional on $L(X, Y)$, although $p \in X^{**}$ and $q \in Y^*$ (or even $q \in Y^{***}$) are extreme points of the respective unit balls. On the other hand it has been observed in [**9**] that $T \to \langle T\mathbf{1}, \delta_k\rangle$ is an extreme functional on $L(CK, CK)$ if k is an isolated point of K or $K = \beta I$ for a discrete set I (the point here is that $\mathbf{1} \notin \operatorname{dent} B_{CK}$ unless K is finite).

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