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ON THE INTERSECTION OF VARIETIES WITH A TOTALLY REAL SUBMANIFOLD

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ABSTRACT. In their work on uniqueness in the Cauchy problem for CR functions, Baouendi and Treves have utilized a condition on a totally real submanifold M in a neighborhood of one of its points p: There should exist a variety X such that the component containing p of $M - (M \cap X)$ has compact closure in M. All real analytic submanifolds satisfy this condition. In this paper, a C^∞ submanifold is constructed which does not. Uniqueness in the corresponding Cauchy problem remains unresolved.

The generic intersection of two submanifolds of $\mathbf{R}^4$, each of dimension two, is a single point. Of course in any particular case other intersections are possible. For instance, the totally real submanifold $\{(z_1, z_2)\colon z_1 \text{ and } z_2 \text{ are real}\}$ of $\mathbf{C}^2$ intersects the complex curve $\{(z_1, z_2)\colon z_1^2 + z_2^2 = 1\}$ in a circle. From this we may see that for any totally real, real analytic surface M^2 in $\mathbf{C}^2$ which contains the origin and for any neighborhood V of the origin there exists a smaller neighborhood U and a function F holomorhic on U such that $M^2 \cap F^{-1}(0)$ contains a closed curve about the origin.

The following condition has been shown by Baouendi and Treves to imply a uniqueness result for CR functions [**BT**]. We state this condition as it arises in the uniqueness result for four-dimensional CR submanifolds in $\mathbf{C}^3$. Here Σ is a totally real two-dimensional manifold in $\mathbf{C}^2$ and p is some point of Σ.

Condition A. Given any sufficiently small neighborhood V of p in Σ there is a function F holomorphic in an open set of $\mathbf{C}^2$ containing V such that $F(p) \neq 0$ and the connected component of p in the set $V \cap \{q\colon F(q) \neq 0\}$ has compact closure contained in V.

Note that if for a particular V such a function F exists, then F also suffices for a somewhat smaller V' and now F is holomorphic in a neighborhood of $\overline{V'}$. Hence to provide a Σ which does not satisfy Condition A we need only show that Σ does not satisfy the somewhat more restrictive condition where "containing V" is replaced by "containing $\overline{V}$."

By the remarks above, Condition A and hence also the uniquenes result in [**BT**] hold when Σ is real analytic. In this paper we construct a Σ of class C^∞ for which Condition A fails. It is not known if the uniqueness result also fails. In our proof that Condition A is not valid for this particular Σ, we use the following two lemmas.

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LEMMA 1. *Let U and U_1 be open sets in $\mathbf{C}^2$ with $\overline{U} \subset U_1$ and let F be a holomorphic function on U_1, F not identically zero on any component of U. Then for each $q \in U \cap F^{-1}(0)$, except for finitely many, there exists an open neighborhood U_q of q and a function F_q holomorphic on U_q such that*

(a) *F_q is either of the form $z_1 - f(z_2)$ or $z_2 - f(z_1)$,*

(b) *$U_q \cap F^{-1}(0) = U_q \cap F_q^{-1}(0)$.*

PROOF. Recall that a point q on the variety $F^{-1}(0)$ is called a regular point if there is a neighborhood V_q of q such that $F^{-1}(0) \cap V_q$ is a complex submanifold of V_q [**GR**]. On some smaller neighborhood U_q, this submanifold must be given either by $z_1 = f(z_2)$ or $z_2 = f(z_1)$. Thus we need show that there can only be a finite number of singular points of $F^{-1}(0) \cap U$. This is true because the singular points form a variety of dimension less than that of $F^{-1}(0)$ [**W**, p. 92] and, as is easily seen, any variety of dimension zero is a discrete set of points.

LEMMA 2. *Let F be a complex valued, real analytic function near the origin in $\mathbf{R}^2$. There exist a neighborhood U of the origin and an integer N such that on each line segment $\{(x,y)\colon y = c\} \cap U$ either F has less than N zeros or F is identically zero.*

This follows from, for instance, the much stronger results in [**H**].

The totally real two-dimensional submanifold Σ of $\mathbf{C}^2$ which we seek will be of the form

$$\Sigma = \{(z_1, z_2)\colon z_1 = x_1,\ z_2 = x_2 + i\phi(x_1)\},$$

where ϕ is a C^∞ function which we now describe. Let $p_j = 1/j$ and let $\{B_{j,k}\}$ be a set of disjoint intervals with $B_{j,k}$ converging to the point p_j as k goes to infinity. Let $\phi \in C^\infty(\mathbf{R})$ be positive in each $B_{j,k}$, zero at each p_j, and with supp $\phi = \overline{\bigcup_{j,k} B_{j,k}}$. Let $S_{j,k} = \{(x_1, x_2)\colon x_1 \in B_{j,k}\}$ and $l_j = \{(x_1, x_2)\colon x_1 = p_j\}$. Then as a function on $\mathbf{R}^2$, supp $\phi = \overline{\bigcup_{j,k} S_{j,k}}$. Note that for any j and any integer N there exists an ε such that $\phi(x) = \mu$ has more than N solutions in $\bigcup_k B_{j,k}$ provided $0 < \mu < \varepsilon$. Also note that Σ is totally real because at any point $p \in \Sigma$ the tangent space $T_p\Sigma$ is spanned by $\partial/\partial x_2$ and $\partial/\partial x_1 + \phi'(x_1)\partial/\partial y_2$, and so $T_p\Sigma \cap J(T_p\Sigma) = \{0\}$ for the complex structure operator J.

THEOREM. *Let V be any neighborhood of the origin in Σ and F any function holomorphic in a open set of $\mathbf{C}^2$ containing $\overline{V}$ with $F(0) \neq 0$. There exists a piecewise smooth curve $\gamma(t)$ in Σ with $\gamma(0) = 0$, $\gamma(1) \in \partial V$, and $F(\gamma(t))$ different from zero for each t in $0 \le t \le 1$.*

PROOF. We may replace V by its connected component containing the origin. Let F be holomorphic on some U_1 and let U be a connected open set with $\overline{V} \subset U \subset \overline{U} \subset U_1$. We apply Lemma 1 to find some j for which l_j does not contain any of the finite number of points excluded in this lemma. We also take j large enough so that $F(x, 0) \neq 0$ for x satisfying $0 \le x \le 1/j$. In particular F cannot be identically zero on the component of $l_j \cap U$ which contains $(1/j, 0)$ and so F can have only a finite number of zeros on the component of $l_j \cap \overline{V}$ which contains this point. If we enlarge V a little then we also have F is different from zero at the two endpoints of this component. (We may enlarge V, since if the theorem holds for some V' with

$\overline{V} \subset V'$, then it clearly also holds for V.) For this choice of j let p be the point $(1/j, 0)$, l the line l_j, and S_k the strip $S_{j,k}$. To summarize, we now have

(a) F has only finitely many zeros on the component of $l \cap V$ which contains p;

(b) near each of these zeros, the zero set of F in $\mathbf{C}^2$ is given by either $z_1 = f(z_2)$ or $z_2 = f(z_1)$; and

(c) $F \neq 0$ at the two boundary points of this component of l.

Since there is a curve in Σ from the origin to p on which F is not zero, it suffices to find such a curve from p to one of the boundary points of this component of l. This curve will proceed along l until it approaches a zero of F. The next lemma is used to avoid this zero along a curve which returns to l beyond this zero. A finite number of such detours then provides the curve $\gamma(t)$ in the theorem. For convenience we introduce new coordinates so that l is the x_2-axis, p is the origin, and I is the line segment $I = \{(0, s) \colon 0 \leq s < 1\}$ which lies in V with endpoint $(0, 1)$ on ∂V. We have F holomorphic in a neighborhood of I; $F(0,0) \neq 0$; $F(0,1) \neq 0$; and F has only a finite number of zeros on I. Let $q = (0, s)$ be any one of these zeros and for some small δ take numbers s_- and s_+ with $s - \delta < s_- < s < s_+ < s + \delta$. Let $q_\pm = (0, s_\pm)$.

LEMMA 3. *If δ is small enough, there exists a piecewise smooth curve $\sigma(t)$ in Σ with $\sigma(0) = q_-$, $\sigma(1) = q_+$, and $F(\sigma(t))$ different from zero for each t in $0 \leq t \leq 1$.*

PROOF. Find an ε such that in the ball $B(q, \varepsilon)$ the zero set of F is given by

(i) $z_1 = f(z_2)$ or

(ii) $z_2 = f(z_1)$.

Now choose ε_1 and δ so small that $B(q_-, \varepsilon_1)$ and $B(q_+, \varepsilon_1)$ are contained in $B(q, \varepsilon)$. Let $\mathcal{D}$ be the "rectangle" in Σ given by

$$\mathcal{D} = \{x_1, x_2 + i\phi(x_1) \colon 0 \leq x_1 \leq \varepsilon_1,\ s_- \leq x_2 \leq s_+\}.$$

Thus $\mathcal{D}$ is bounded by a segment of l, a line segment parallel to l, and by curves passing through q_- and q_+. Note that $\mathcal{D} \subset B(q, \varepsilon)$ and so one of the forms (i) or (ii) is valid throughout $\mathcal{D}$. Also note that as long as ε_1 is small enough we have that F is nonzero on the top and bottom boundaries of $\mathcal{D}$. Thus, to prove the lemma, we need only find a segment in $\mathcal{D}$ parallel to l along which F is never zero.

The proof that such a segment exists is divided into two cases corresponding to the equations $z_1 = f(z_2)$ and $z_2 = f(z_1)$ for the zero set.

Case (i). The zero set of F is given by $z_1 = f(z_2)$. Let $v(x_2, y_2) = \operatorname{Im} f(x_2 + iy_2)$. We claim that v is not identically zero. This is so because if it were, then $f(z_2)$ would be a constant but $f(s) = 0$ and $f(s_-) \neq 0$. Thus there exists a sequence of real numbers $\mu_k \to 0$ such that $v(x_2, \mu_k)$ is not identically zero on the interval $s_- \leq x_2 \leq s_+$. Lemma 2 then guarantees that for some integer N each equation $v(x_2, \mu_k) = 0$, for k large, can have no more than N solutions $x_2 = x_2(\mu_k)$ in this interval. (The application of this lemma might necessitate shrinking ε, and hence also ε_1 and δ.) Clearly we may take each μ_k to be positive. We want to show that F is everywhere different from zero for some segment in $\mathcal{D}$. We reason by contradiction and so start with the assumption that for each x_1 there is some x_2 in $s_- \leq x_2 \leq s_+$ such that F has a zero at the point $(x_1, x_2 + i\phi(x_1))$. Thus for each x_1 in $0 \leq x_1 \leq \varepsilon_1$ there is at least one x_2 in $s_- \leq x_2 \leq s_+$ with $x_1 = f(x_2 + i\phi(x_1))$. For each μ let $N(\mu)$ denote the number of solutions to $\phi(x_1) = \mu$ in $0 \leq x_1 \leq \varepsilon_1$.

As we have already noted, $N(\mu) \to \infty$ as $\mu \to 0$ with $\mu > 0$. Thus the equation $x_1 = f(x_2 + i\mu)$ has $N(\mu)$ solution pairs (x_1, x_2). Further, no two solution pairs with different values of x_1 can have the same x_2. Thus along the line segment in the z_2-plane given by $s_- \le \operatorname{Re} z_2 \le s_+$ and $\operatorname{Im} z_2 = \mu$, the holomorphic function f is real for at least $N(\mu)$ points. But then $N(\mu) \to \infty$ contradicts the existence of the bound N for solutions to $\operatorname{Im} f(x_2 + i\mu_k) = 0$.

Case (ii). The zero set of F is given by $z_2 = f(z_1)$. If F does not have a zero on some segment $\{(x_1, x_2) \colon x_1 = c \notin \operatorname{supp} \phi,\ s_- \le x_2 \le s_+\}$, then we are done. So we now show that if F does have a zero on each such segment then F cannot have any zeros in the interior of the support of ϕ and thus any segment in this set provides the desired curve. So let us assume that for each $x_1 \notin \operatorname{supp} \phi$ there is some x_2 with $x_2 = f(x_1)$. Thus f is real valued on an interval of the real axis. Hence it is real valued on the entire piece of the x_1-axis in $B(q, \varepsilon)$. Thus $x_2 + i\phi(x_1) = f(x_1)$ can be satisfied only when $\phi(x_1) = 0$. Hence f has no zeros in $\{(x_1, x_2 + i\phi(x_1)) \colon \phi(x_1) \neq 0,\ s_- \le x_2 \le s_+\}$.

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