ON THE INTERSECTION OF VARIETIES WITH A TOTALLY REAL SUBMANIFOLD
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ABSTRACT. In their work on uniqueness in the Cauchy problem for CR functions, Baouendi and Tréves have utilized a condition on a totally real submanifold $M$ in a neighborhood of one of its points $p$: There should exist a variety $X$ such that the component containing $p$ of $M - (M \cap X)$ has compact closure in $M$. All real analytic submanifolds satisfy this condition. In this paper, a $C^\infty$ submanifold is constructed which does not. Uniqueness in the corresponding Cauchy problem remains unresolved.

The generic intersection of two submanifolds of $\mathbb{R}^4$, each of dimension two, is a single point. Of course in any particular case other intersections are possible. For instance, the totally real submanifold $\{ (z_1, z_2) : z_1$ and $z_2$ are real $\}$ of $\mathbb{C}^2$ intersects the complex curve $\{ (z_1, z_2) : z_1^2 + z_2^2 = 1 \}$ in a circle. From this we may see that for any totally real, real analytic surface $M^2$ in $\mathbb{C}^2$ which contains the origin and for any neighborhood $V$ of the origin there exists a smaller neighborhood $U$ and a function $F$ holomorphic on $U$ such that $M^2 \cap F^{-1}(0)$ contains a closed curve about the origin.

The following condition has been shown by Baouendi and Tréves to imply a uniqueness result for CR functions [BT]. We state this condition as it arises in the uniqueness result for four-dimensional CR submanifolds in $\mathbb{C}^3$. Here $\Sigma$ is a totally real two-dimensional manifold in $\mathbb{C}^2$ and $p$ is some point of $\Sigma$.

Condition A. Given any sufficiently small neighborhood $V$ of $p$ in $\Sigma$ there is a function $F$ holomorphic in an open set of $\mathbb{C}^2$ containing $V$ such that $F(p) \neq 0$ and the connected component of $p$ in the set $V \cap \{ q : F(q) \neq 0 \}$ has compact closure contained in $V$.

Note that if for a particular $V$ such a function $F$ exists, then $F$ also suffices for a somewhat smaller $V'$ and now $F$ is holomorphic in a neighborhood of $V'$. Hence to provide a $\Sigma$ which does not satisfy Condition A we need only show that $\Sigma$ does not satisfy the somewhat more restrictive condition where “containing $V$” is replaced by “containing $V'$.”

By the remarks above, Condition A and hence also the uniqueness result in [BT] hold when $\Sigma$ is real analytic. In this paper we construct a $\Sigma$ of class $C^\infty$ for which Condition A fails. It is not known if the uniqueness result also fails. In our proof that Condition A is not valid for this particular $\Sigma$, we use the following two lemmas.
LEMMA 1. Let $U$ and $U_1$ be open sets in $\mathbb{C}^2$ with $\overline{U} \subset U_1$ and let $F$ be a holomorphic function on $U_1$, $F$ not identically zero on any component of $U$. Then for each $q \in U \cap F^{-1}(0)$, except for finitely many, there exists an open neighborhood $U_q$ of $q$ and a function $F_q$ holomorphic on $U_q$ such that

(a) $F_q$ is either of the form $z_1 - f(z_2)$ or $z_2 - f(z_1)$,

(b) $U_q \cap F^{-1}(0) = U_q \cap F_q^{-1}(0)$.

PROOF. Recall that a point $q$ on the variety $F^{-1}(0)$ is called a regular point if there is a neighborhood $V_q$ of $q$ such that $F^{-1}(0) \cap V_q$ is a complex submanifold of $V_q$ [GR]. On some smaller neighborhood $U_q$, this submanifold must be given either by $z_1 = f(z_2)$ or $z_2 = f(z_1)$. Thus we need show that there can only be a finite number of singular points of $F^{-1}(0) \cap U$. This is true because the singular points form a variety of dimension less than that of $F^{-1}(0)$ [W, p. 92] and, as is easily seen, any variety of dimension zero is a discrete set of points.

LEMMA 2. Let $F$ be a complex valued, real analytic function near the origin in $\mathbb{R}^2$. There exist a neighborhood $U$ of the origin and an integer $N$ such that on each line segment $\{(x,y): y = c\} \cap U$ either $F$ has less than $N$ zeros or $F$ is identically zero.

This follows from, for instance, the much stronger results in [H].

The totally real two-dimensional submanifold $\Sigma$ of $\mathbb{C}^2$ which we seek will be of the form

$$\Sigma = \{(z_1, z_2): z_1 = x_1, \quad z_2 = x_2 + i\phi(x_1)\},$$

where $\phi$ is a $C^\infty$ function which we now describe. Let $p_j = 1/j$ and let $\{B_{j,k}\}$ be a set of disjoint intervals with $B_{j,k}$ converging to the point $p_j$ as $k$ goes to infinity. Let $\phi \in C^\infty(\mathbb{R})$ be positive in each $B_{j,k}$, zero at each $p_j$, and with $\text{supp} \phi = \bigcup_{j,k} B_{j,k}$.

Let $S_{j,k} = \{(x_1, x_2): x_1 \in B_{j,k}\}$ and $l_j = \{(x_1, x_2): x_1 = p_j\}$. Then as a function on $\mathbb{R}^2$, $\text{supp} \phi = \bigcup_{j,k} S_{j,k}$. Note that for any $j$ and any integer $N$ there exists an $\varepsilon$ such that $\phi(x) = \mu$ has more than $N$ solutions in $\bigcup_k B_{j,k}$ provided $0 < \mu < \varepsilon$.

Also note that $\Sigma$ is totally real because at any point $p \in \Sigma$ the tangent space $T_p \Sigma$ is spanned by $\partial / \partial x_2$ and $\partial / \partial x_1 + \phi'(x_1) \partial / \partial y_2$, and so $T_p \Sigma \cap J(T_p \Sigma) = \{0\}$ for the complex structure operator $J$.

THEOREM. Let $V$ be any neighborhood of the origin in $\Sigma$ and $F$ any function holomorphic in a open set of $\mathbb{C}^2$ containing $V$ with $F(0) \neq 0$. There exists a piecewise smooth curve $\gamma(t)$ in $\Sigma$ with $\gamma(0) = 0$, $\gamma(1) \in \partial V$, and $F(\gamma(t))$ different from zero for each $t$ in $0 \leq t \leq 1$.

PROOF. We may replace $V$ by its connected component containing the origin. Let $F$ be holomorphic on some $U_1$ and let $U$ be a connected open set with $\overline{V} \subset U \subset U_1$. We apply Lemma 1 to find some $j$ for which $l_j$ does not contain any of the finite number of points excluded in this lemma. We also take $j$ large enough so that $F(x,0) \neq 0$ for $x$ satisfying $0 \leq x \leq 1/j$. In particular $F$ cannot be identically zero on the component of $l_j \cap U$ which contains $(1/j,0)$ and so $F$ can have only a finite number of zeros on the component of $l_j \cap \overline{V}$ which contains this point. If we enlarge $V$ a little then we also have $F$ is different from zero at the two endpoints of this component. (We may enlarge $V$, since if the theorem holds for some $V'$ with
V \subset V'$, then it clearly also holds for $V_j$.) For this choice of $j$ let $p$ be the point $(1/j, 0)$, $l$ the line $l_j$, and $S_k$ the strip $S_{j,k}$. To summarize, we now have

(a) $F$ has only finitely many zeros on the component of $l \cap V$ which contains $p$;
(b) near each of these zeros, the zero set of $F$ in $C^2$ is given by either $z_1 = f(z_2)$ or $z_2 = f(z_1)$; and
(c) $F \neq 0$ at the two boundary points of this component of $l$.

Since there is a curve in $\Sigma$ from the origin to $p$ on which $F$ is not zero, it suffices to find such a curve from $p$ to one of the boundary points of this component of $l$. This curve will proceed along $l$ until it approaches a zero of $F$. The next lemma is used to avoid this zero along a curve which returns to $l$ beyond this zero. A finite number of such detours then provides the curve $\gamma(t)$ in the theorem. For convenience we introduce new coordinates so that $l$ is the $x_2$-axis, $p$ is the origin, and $I$ is the line segment $I = \{(0, s): 0 \leq s < 1\}$ which lies in $V$ with endpoint $(0, 1)$ on $\partial V$. We have $F$ holomorphic in a neighborhood of $I$; $F(0,0) \neq 0$; $F(0,1) \neq 0$; and $F$ has only a finite number of zeros on $I$. Let $q = (0, s)$ be any one of these zeros and for some small $\delta$ take numbers $s_-$ and $s_+$ with $s-\delta < s_- < s < s_+ < s + \delta$. Let $q_\pm = (0,s_\pm)$.

**Lemma 3.** If $\delta$ is small enough, there exists a piecewise smooth curve $\sigma(t)$ in $\Sigma$ with $\sigma(0) = q_-$, $\sigma(1) = q_+$, and $F(\sigma(t))$ different from zero for each $t$ in $0 \leq t \leq 1$.

**Proof.** Find an $\varepsilon$ such that in the ball $B(q,\varepsilon)$ the zero set of $F$ is given by

(i) $z_1 = f(z_2)$ or
(ii) $z_2 = f(z_1)$.

Now choose $\delta_1$ and $\delta$ so small that $B(q_-,\delta_1)$ and $B(q_+,\delta_1)$ are contained in $B(q,\varepsilon)$. Let $D$ be the “rectangle” in $\Sigma$ given by

$$D = \{x_1, x_2 + i\phi(x_1): 0 < x_1 < \delta_1, s_- < x_2 < s_+\}.$$ 

Thus $D$ is bounded by a segment of $l$, a line segment parallel to $l$, and by curves passing through $q_-$ and $q_+$. Note that $D \subset B(q,\varepsilon)$ and so one of the forms (i) or (ii) is valid throughout $D$. Also note that as long as $\varepsilon_1$ is small enough we have that $F$ is nonzero on the top and bottom boundaries of $D$. Thus, to prove the lemma, we need only find a segment in $D$ parallel to $l$ along which $F$ is never zero.

The proof that such a segment exists is divided into two cases corresponding to the equations $z_1 = f(z_2)$ and $z_2 = f(z_1)$ for the zero set.

Case (i). The zero set of $F$ is given by $z_1 = f(z_2)$. Let $v(x_2,y_2) = \text{Im } f(x_2 + iy_2)$. We claim that $v$ is not identically zero. This is so because if it were, then $f(z_2)$ would be a constant but $f(s) = 0$ and $f(s_-) \neq 0$. Thus there exists a sequence of real numbers $\mu_k \to 0$ such that $v(x_2,\mu_k)$ is not identically zero on the interval $s_- \leq x_2 \leq s_+$. Lemma 2 then guarantees that for some integer $N$ each equation $v(x_2,\mu_k) = 0$, for $k$ large, can have no more than $N$ solutions $x_2 = x_2(\mu_k)$ in this interval. (The application of this lemma might necessitate shrinking $\varepsilon$, and hence also $\varepsilon_1$ and $\delta$.) Clearly we may take each $\mu_k$ to be positive. We want to show that $F$ is everywhere different from zero for some segment in $D$. We reason by contradiction and so start with the assumption that for each $x_1$ there is some $x_2$ in $s_- \leq x_2 \leq s_+$ such that $F$ has a zero at the point $(x_1, x_2 + i\phi(x_1))$. Thus for each $x_1$ in $0 \leq x_1 \leq \varepsilon_1$ there is at least one $x_2$ in $s_- \leq x_2 \leq s_+$ with $x_1 = f(x_2 + i\phi(x_1))$. For each $\mu$ let $N(\mu)$ denote the number of solutions to $\phi(x_1) = \mu$ in $0 \leq x_1 \leq \varepsilon_1$. 


As we have already noted, \( N(\mu) \to \infty \) as \( \mu \to 0 \) with \( \mu > 0 \). Thus the equation \( x_1 = f(x_2 + i\mu) \) has \( N(\mu) \) solution pairs \((x_1, x_2)\). Further, no two solution pairs with different values of \( x_1 \) can have the same \( x_2 \). Thus along the line segment in the \( z_2 \)-plane given by \( s_- \leq \Re z_2 \leq s_+ \) and \( \Im z_2 = \mu \), the holomorphic function \( f \) is real for at least \( N(\mu) \) points. But then \( N(\mu) \to \infty \) contradicts the existence of the bound \( N \) for solutions to \( \Im f(x_2 + i\mu) = 0 \).

**Case (ii).** The zero set of \( F \) is given by \( z_2 = f(z_1) \). If \( F \) does not have a zero on some segment \( \{(x_1, x_2) : x_1 = c \notin \text{supp } \phi, s_- \leq x_2 \leq s_+\} \), then we are done. So we now show that if \( F \) does have a zero on each such segment then \( F \) cannot have any zeros in the interior of the support of \( \phi \) and thus any segment in this set provides the desired curve. So let us assume that for each \( x_1 \notin \text{supp } \phi \) there is some \( x_2 \) with \( x_2 = f(x_1) \). Thus \( f \) is real valued on an interval of the real axis. Hence it is real valued on the entire piece of the \( x_1 \)-axis in \( B(q, \varepsilon) \). Thus \( x_2 + i\phi(x_1) = f(x_1) \) can be satisfied only when \( \phi(x_1) = 0 \). Hence \( f \) has no zeros in \( \{(x_1, x_2 + i\phi(x_1)) : \phi(x_1) \neq 0, s_- \leq x_2 \leq s_+\} \).

REFERENCES


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