

## ORTHOGONAL COMPLEX STRUCTURES ON $S^6$

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**ABSTRACT.** A complex structure on the six-sphere is called *orthogonal* if the standard metric is Hermitian with respect to it. While such structures locally exist in profusion, there is no such complex structure on the entire sphere.

An almost-complex structure  $J$  on a smooth manifold  $M$  (of necessarily even dimension) is an endomorphism  $J: TM \rightarrow TM$  of the tangent bundle such that  $J^2 = -1$ ; such a structure identifies  $TM$  with a *complex* vector bundle by letting scalar multiplication by  $\sqrt{-1}$  be defined to be the endomorphism  $J$ . It is known [1] that the only spheres admitting such structures are  $S^2$  and  $S^6$ ; such structures are naturally defined if one takes as models of  $S^2$  and  $S^6$ , respectively, the sets of quaternions and octonians with square  $-1$ , allowing one to define the tensor  $J$  at  $x$  simply to be multiplication by  $x$ .

The above structure on  $S^6$  is not integrable in the sense that one cannot find local charts in which  $J$  becomes multiplication by  $\sqrt{-1}$  in  $\mathbf{C}^3$ . In fact it would be a minor disaster if  $S^6$  were to admit the structure of a complex 3-manifold in the following sense: If we then blew up a point we would obtain a complex 3-manifold diffeomorphic to  $\mathbf{CP}_3$  which would not be biholomorphically equivalent to  $\mathbf{CP}_3$ . (Notice that we *can* carry out this process on the level of almost-complex structures, thereby obtaining an "exotic" almost-complex structure on  $\mathbf{CP}_3$  with  $c_1^3 = -8$ , in contrast to the usual structure with  $c_1^3 = 64$ .) This would stand in marked contrast to the result [2] that  $\mathbf{CP}_3$  has only one isomorphism class of Kähler complex structures.

In this note, we restrict our attention to a subclass of almost-complex structures on the six-sphere, namely those which act on the tangent space as isometries with respect to the usual "round" metric; such an almost complex structure will be called *orthogonal*. Thus, the standard example cited above is an orthogonal almost-complex structure. We will show that no globally defined orthogonal almost-complex structure is integrable.

Let  $J$  be any almost-complex structure on  $U \subset S^6$ , and consider the complex vector bundle  $T^{0,1} \rightarrow U$  defined as the  $-i$  eigenspace of  $J$ :

$$T^{0,1} = \{v \in (TS^6) \otimes \mathbf{C} \mid Jv = -iv\}.$$

Since  $T_x S^6 \subset \mathbf{R}^7$  for every  $x \in S^6$ ,  $T_x^{0,1} \subset \mathbf{C}^7$ , and we have a tautological map

$$\tau: U \rightarrow G_3(\mathbf{C}^7)$$

into the Grassmannian of 3-planes in  $\mathbf{C}^7$ . If  $\mathcal{G} \subset G_3(\mathbf{C}^7)$  is the open subset consisting of planes  $P$  for which  $P \cap \bar{P} = \{0\}$ ,  $\tau(S^6) \subset \mathcal{G}$ . But if we define a

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projection  $\pi: \mathcal{G} \rightarrow S^6$  by assigning to a complex 3-plane  $P$  the oriented normal of its real projection  $(P + \bar{P}) \cap \mathbf{R}^7$ , it follows that  $\pi\tau: U \rightarrow U$  is the identity, so  $\tau$  is necessarily an embedding.

LEMMA. *Let  $J$  be orthogonal with respect to the usual metric on  $S^6 \subset \mathbf{R}^7$ . If  $J$  is integrable then  $\tau$  is a holomorphic map.*

PROOF. If  $J$  is orthogonal and  $V$  and  $W$  are two elements of  $T_x^{0,1}$ , one has

$$g(V, W) = g(JV, JW) = g(-iV, -iW) = -g(V, W)$$

so that  $g(V, W) = 0$ , where  $g$  is the usual metric on  $\mathbf{R}^7$  extended to  $\mathbf{C}^7$  by complex linearity in both factors. Let  $z^\alpha$ ,  $\alpha = 1, 2, 3$ , be local holomorphic coordinates on some region of  $S^6$  and let  $X_\alpha$  be complex vector fields on  $S^6$  determined by the equations

$$g(X_\alpha, Y) = \langle dz^\alpha, Y \rangle$$

for all complex vector fields  $Y$  on  $S^6$ . Then  $X_\alpha$  is a section of  $T^{0,1}$  because  $\langle dz^\alpha, V \rangle = 0$  for all  $V \in T^{0,1}$ . Introducing the summation convention, using the Euclidean metric to raise and lower Latin indices, and letting  $\nabla$  be the Levi-Civita connection on  $S^6$ , we have

$$\begin{aligned} X_\alpha^a \nabla_a (X_\beta)_b &= X_\alpha^a \nabla_a (dz^\beta)_b = X_\alpha^a \nabla_b (dz^\beta)_a \\ &= -(dz^\beta)_a \nabla_b X_\alpha^a = -(X_\beta)_a \nabla_b (dz^\alpha)^a \\ &= -X_\beta^a \nabla_b (dz^\alpha)_a = -X_\beta^a \nabla_a (dz^\alpha)_b = -X_\beta^a \nabla_a X_b^\alpha, \end{aligned}$$

so that  $\nabla_{X_\alpha} X_\beta = -\nabla_{X_\beta} X_\alpha$ , and

$$[X_\alpha, X_\beta] = \nabla_{X_\alpha} X_\beta - \nabla_{X_\beta} X_\alpha = 2\nabla_{X_\alpha} X_\beta.$$

But the integrability condition implies that  $[X_\alpha, X_\beta]$  is a section of  $T^{0,1}$ . Hence

$$\nabla_{X_\alpha} X_\beta \equiv 0 \pmod{\text{span}\{X_\beta\}}.$$

But the Levi-Civita connection  $\nabla$  on the unit sphere is related to the flat connection  $D$  on  $R^7$  by  $\nabla_W V = D_W V + g(V, W)N$ , where  $N$  is the unit outward-pointing normal vector field of  $S^6$ ; hence

$$\begin{aligned} D_{X_\alpha} X_\beta &= \nabla_{X_\alpha} X_\beta - g(X_\alpha, X_\beta)N = \nabla_{X_\alpha} X_\beta \\ &= \frac{1}{2}[X_\alpha, X_\beta] \equiv 0 \pmod{\text{span}\{X_\gamma\}}. \end{aligned}$$

Since  $\text{Hom}(T_x^{0,1}, \mathbf{C}^7/T_x^{0,1})$  is precisely the holomorphic tangent space of  $G_3(\mathbf{C}^7)$  at  $T_x^{0,1}$  the above calculation shows that all the  $(0, 1)$ -components of the derivative of  $\tau$  vanish, and  $\tau$  is holomorphic. Q.E.D.

As corollary we get the following result:

**THEOREM.**  *$S^6$  has no integrable orthogonal complex structure.*

PROOF. Suppose it did. Then  $\tau$  would be an embedding of  $S^6$  in  $G_3(\mathbf{C}^7)$  as a complex manifold. But Grassmannians are Kähler manifolds, so this would give  $S^6$  a Kähler structure. But this is impossible because  $H^2(S^6) = 0$ . Q.E.D.

REMARK. For the lemma the dimension of  $S^6$  was inessential; if we applied the same technique to the usual orthogonal complex structure on  $S^2$  we would obtain a holomorphic map  $\tau: S^2 \rightarrow G_2(\mathbf{C}^3) = \mathbf{CP}_2$ , thereby realizing the 2-sphere as a

nonsingular plane conic. Of course, the contradiction of the theorem does not occur in this case because  $H^2(S^2) \neq 0$ .

It remains to be seen whether techniques of this kind may be generalized to the nonorthogonal case.

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