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ORTHOGONAL COMPLEX STRUCTURES ON S^6

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ABSTRACT. A complex structure on the six-sphere is called *orthogonal* if the standard metric is Hermitian with respect to it. While such structures locally exist in profusion, there is no such complex structure on the entire sphere.

An almost-complex structure J on a smooth manifold M (of necessarily even dimension) is an endomorphism $J: TM \to TM$ of the tangent bundle such that $J^2 = -1$; such a structure identifies TM with a *complex* vector bundle by letting scalar multiplication by $\sqrt{-1}$ be defined to be the endomorphism J. It is known [1] that the only spheres admitting such structures are S^2 and S^6; such structures are naturally defined if one takes as models of S^2 and S^6, respectively, the sets of quaternions and octonians with square -1, allowing one to define the tensor J at x simply to be multiplication by x.

The above structure on S^6 is not integrable in the sense that one cannot find local charts in which J becomes multiplication by $\sqrt{-1}$ in $\mathbf{C}^3$. In fact it would be a minor disaster if S^6 *were* to admit the structure of a complex 3-manifold in the following sense: If we then blew up a point we would obtain a complex 3-manifold diffeomorphic to $\mathbf{CP}_3$ which would not be biholomorphically equivalent to $\mathbf{CP}_3$. (Notice that we *can* carry out this process on the level of almost-complex structures, thereby obtaining an "exotic" almost-complex structure on $\mathbf{CP}_3$ with $c_1^3 = -8$, in contrast to the usual structure with $c_1^3 = 64$.) This would stand in marked contrast to the result [2] that $\mathbf{CP}_3$ has only one isomorphism class of Kähler complex structures.

In this note, we restrict our attention to a subclass of almost-complex structures on the six-sphere, namely those which act on the tangent space as isometries with respect to the usual "round" metric; such an almost complex structure will be called *orthogonal.* Thus, the standard example cited above is an orthogonal almost-complex structure. We will show that no globally defined orthogonal almost-complex structure is integrable.

Let J be any almost-complex structure on $U \subset S^6$, and consider the complex vector bundle $T^{0,1} \to U$ defined as the $-i$ eigenspace of J:

$$T^{0,1} = \{v \in (TS^6) \otimes \mathbf{C} \mid Jv = -iv\}.$$

Since $T_xS^6 \subset \mathbf{R}^7$ for every $x \in S^6$, $T_x^{0,1} \subset \mathbf{C}^7$, and we have a tautological map

$$\tau: U \to G_3(\mathbf{C}^7)$$

into the Grassmannian of 3-planes in $\mathbf{C}^7$. If $\mathcal{G} \subset G_3(\mathbf{C}^7)$ is the open subset consisting of planes P for which $P \cap \overline{P} = \{0\}$, $\tau(S^6) \subset \mathcal{G}$. But if we define a

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projection $\pi\colon \mathcal{G} \to S^6$ by assigning to a complex 3-plane P the oriented normal of its real projection $(P+\overline{P}) \cap \mathbf{R}^7$, it follows that $\pi\tau\colon U \to U$ is the identity, so τ is necessarily an embedding.

LEMMA. *Let J be orthogonal with respect to the usual metric on $S^6 \subset \mathbf{R}^7$. If J is integrable then τ is a holomorphic map.*

PROOF. If J is orthogonal and V and W are two elements of $T_x^{0,1}$, one has

$$g(V,W) = g(JV,JW) = g(-iV,-iW) = -g(V,W)$$

so that $g(V,W)=0$, where g is the usual metric on $\mathbf{R}^7$ extended to $\mathbf{C}^7$ by complex linearity in both factors. Let z^α, $\alpha = 1,2,3$, be local holomorphic coordinates on some region of S^6 and let X_α be complex vector fields on S^6 determined by the equations

$$g(X_\alpha, Y) = \langle dz^\alpha, Y\rangle$$

for all complex vector fields Y on S^6. Then X_α is a section of $T^{0,1}$ because $\langle dz^\alpha, V\rangle = 0$ for all $V \in T^{0,1}$. Introducing the summation convention, using the Euclidean metric to raise and lower Latin indices, and letting ∇ be the Levi-Civita connection on S^6, we have

$$\begin{aligned} X_\alpha^a \nabla_a (X_\beta)_b &= X_\alpha^a \nabla_a (dz^\beta)_b = X_\alpha^a \nabla_b (dz^\beta)_a \\ &= -(dz^\beta)_a \nabla_b X_\alpha^a = -(X_\beta)_a \nabla_b (dz^\alpha)^a \\ &= -X_\beta^a \nabla_b (dz^\alpha)_a = -X_\beta^a \nabla_a (dz^\alpha)_b = -X_\beta^a \nabla_a X_b^\alpha, \end{aligned}$$

so that $\nabla_{X_\alpha} X_\beta = -\nabla_{X_\beta} X_\alpha$, and

$$[X_\alpha, X_\beta] = \nabla_{X_\alpha} X_\beta - \nabla_{X_\beta} X_\alpha = 2\nabla_{X_\alpha} X_\beta.$$

But the integrability condition imples that $[X_\alpha, X_\beta]$ is a section of $T^{0,1}$. Hence

$$\nabla_{X_\alpha} X_\beta \equiv 0 \mod \operatorname{span}\{X_\beta\}.$$

But the Levi-Civita connection ∇ on the unit sphere is related to the flat connection D on R^7 by $\nabla_W V = D_W V + g(V,W)N$, where N is the unit outward-pointing normal vector field of S^6; hence

$$\begin{aligned} D_{X_\alpha} X_\beta &= \nabla_{X_\alpha} X_\beta - g(X_\alpha, X_\beta)N = \nabla_{X_\alpha} X_\beta \\ &= \tfrac{1}{2}[X_\alpha, X_\beta] \equiv 0 \mod \operatorname{span}\{X_\gamma\}. \end{aligned}$$

Since $\operatorname{Hom}(T_x^{0,1}, \mathbf{C}^7/T_x^{0,1})$ is precisely the holomorphic tangent space of $G_3(\mathbf{C}^7)$ at $T_x^{0,1}$ the above calculation shows that all the $(0,1)$-components of the derivative of τ vanish, and τ is holomorphic. Q.E.D.

As corollary we get the following result:

THEOREM. *S^6 has no integrable orthogonal complex structure.*

PROOF. Suppose it did. Then τ would be an embedding of S^6 in $G_3(\mathbf{C}^7)$ as a complex manifold. But Grassmannians are Kähler manifolds, so this would give S^6 a Kähler structure. But this is impossible because $H^2(S^6) = 0$. Q.E.D.

REMARK. For the lemma the dimension of S^6 was inessential; if we applied the same technique to the usual orthogonal complex structure on S^2 we would obtain a holomorphic map $\tau\colon S^2 \to G_2(\mathbf{C}^3) = \mathbf{CP}_2$, thereby realizing the 2-sphere as a nonsingular plane conic. Of course, the contradiction of the theorem does not occur in this case because $H^2(S^2) \neq 0$.

It remains to be seen whether techniques of this kind may be generalized to the nonorthogonal case.

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