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AN ADJOINT REPRESENTATION FOR POLYNOMIAL ALGEBRAS

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ABSTRACT. This paper shows that a graded polynomial algebra over F_2 with Steenrod algebra action possesses an analog of the adjoint representation for the cohomology of the classifying space of a compact connected Lie group.

1. Introduction. Let $R = F_2[z_1, z_2, \ldots, z_n]$ be a graded polynomial algebra over the field with two elements and, in addition, suppose that R has an action of the mod 2 Steenrod algebra A for which it is an unstable algebra over A. This condition means that R has the formal algebraic properties satisfied by the mod 2 cohomology of a space. Specifically, there are homomorphisms $\mathrm{Sq}^i\colon R^j \to R^{j+i}$ for $i \geq 0$ satisfying

$$(1)\quad \text{(Cartan formula)}\qquad \mathrm{Sq}^i(xy) = \sum_{j=0}^{i} \mathrm{Sq}^j x \cdot \mathrm{Sq}^{i-j} y,$$

and

$$(2)\quad \text{(Unstability)}\qquad \mathrm{Sq}^i x = \begin{cases} x & \text{if } i = 0, \\ x^2 & \text{if } i = \dim x, \\ 0 & \text{if } i > \dim x. \end{cases}$$

It is convenient to let $\mathrm{Sq} = 1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \cdots$ be the total Steenrod square, which is then an algebra automorphism of R. As examples we have:

(1) Let $R_0 = F_2[x_1, \ldots, x_n]$ with $\dim x_i = 1$. The action of A is then uniquely described by $\mathrm{Sq}\, x_i = x_i + x_i^2$, and R_0 is isomorphic to $H^*(BZ_2^n; Z_2)$, the mod 2 cohomology of the classifying space for the group Z_2^n.

(2) Let $R_1 = (F_2[x_1, \ldots, x_n])^{\Sigma_n}$ be the ring of invariants of the symmetric group acting to permute the x_i. Then $R_1 = F_2[w_1, \ldots, w_n]$, where w_i is the ith elementary symmetric function of the x_j. Then R_1 is isomorphic to $H^*(BO(n); Z_2)$, where $O(n)$ is the orthogonal group, with w_i being called the ith universal Stiefel-Whitney class.

(3) Let $R_2 = D_n = (F_2[x_1, \ldots, x_n])^{\mathrm{GL}_n F_2}$ be the ring of invariants of the general linear group $\mathrm{GL}_n F_2$, considered as the linear transformations of the span of $x_1, \ldots, x_n$. Then $D_n = F_2[c_{n'n-1}, \ldots, c_{n'0}]$, where $\dim c_{n'i} = 2^n - 2^i$, is called the nth Dickson algebra. Except for small values of n it is not the mod 2 cohomology of a space.

The prototypical example is, of course, given by $R = H^*(BG; Z_2)$, which is the mod 2 cohomology of the classifying space of a compact Lie group. For general G the cohomology is not polynomial, but we are considering the polynomial situation. In this situation, with $R = F_2[z_1, \ldots, z_n]$, n is the mod 2 rank of G (see [**Q**]) and

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$m = \sum_1^n (\dim z_i - 1)$ is the dimension of G. In analogy we can call n and m the *rank* and *dimension* of R.

When G is a compact Lie group, we have the adjoint representation giving a homomorphism $\mathrm{Ad}\colon G \to O(m)$. This induces a map $B\,\mathrm{Ad}\colon BG \to BO(m)$, with an induced homomorphism $H^*(BO(m); Z_2) \to H^*(BG; Z_2)$ or equivalently a homomorphism

$$\mathrm{Ad}\colon F_2[w_1, w_2, \ldots, w_m] \to R.$$

The objective of this paper is to show that such a homomorphism actually arises directly from the algebraic structure of R. In particular, a polynomial algebra which is an unstable algebra over the mod 2 Steenrod algebra has a homomorphism

$$\mathrm{Ad}\colon F_2[w_1, \ldots, w_m] \to R = F_2[z_1, \ldots, z_n]$$

which is an analogue of the adjoint representation.

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2. A definition. If G is a compact Lie group, we have the adjoint representation $\mathrm{Ad}\colon G \to O(m)$ and hence an induced homomorphism $\mathrm{Ad}\colon H^*BO(m) \to H^*BG$, where we let $H^*BG = H^*(BG; Z_2)$. Of course, one way to obtain this homomorphism is to consider the representation Ad as giving a vector bundle $EG \times_G R^m \to EG/G = BG$. This vector bundle has a Thom space $T\,\mathrm{Ad}_G$, and the reduced mod 2 cohomology of that Thom space $H^*T\,\mathrm{Ad}_G$ is a free H^*BG module of rank one, with generator a Thom class $U \in H^m T\,\mathrm{Ad}_G$. Then $\mathrm{Sq}\, U = (1 + w_1 + \cdots + w_m)U$, where the class w_i is obtained by writing $\mathrm{Sq}^i U$ as a multiple of U.

More generally, a homomorphism $H^*BO(m) \to R$ always arises from a *Thom module* over R, i.e. a module over the semitensor product $r \odot A$ that is free of rank one as an R-module (Handel [**H**]). We are then asking for a Thom module for R analogous to the adjoint Thom module $H^*T\,\mathrm{Ad}_G$ for H^*BG.

THEOREM. *Let R be a polynomial algebra of rank n and dimension m over the* mod 2 *Steenrod algebra. Then there is a Thom module* Ad_R *over R with the following properties*:

(a) *If $R = H^*BG$, where G is either a compact connected Lie group or $O(n)$, then* $\mathrm{Ad}_R \cong H^*T\,\mathrm{Ad}_G$.

(b) $\dim \mathrm{Ad}_R \leq m$; *i.e.* $w_i = 0$ *if* $i > m$,

(c) *If Q is another polynomial A-algebra,* $\mathrm{Ad}_{Q\otimes R} = \mathrm{Ad}_Q \otimes \mathrm{Ad}_R$.

(d) Ad_R *is trivial* (*i.e.* $w_i = 0$ *for* $i > 0$) *if and only if R is abelian.*

(e) *If T is abelian and $R \to T$ is a nonsingular embedding with Jacobian determinant J, then* $\mathrm{Ad}_R \cong R \cdot J \subset T$.

REMARKS. The meaning of (d) and (e) will be explained in the course of the proof. The groups other than $O(n)$ to which (a) applies are $SO(n)$, $U(n)$, $SU(n)$, $\mathrm{Sp}(n)$, $\mathrm{Spin}(n)$ for $n \leq 9$, G_2, F_4, and products of these (Borel [**B**]). While no details will be given, there are mod p analogues when p is odd prime.

To define Ad_R we make use of the Hochschild homology H_*R, which is defined to be $\mathrm{Tor}^{R\otimes R}(R, R)$ (see [**C-E**]). By standard homological methods H_*R becomes a commutative (in the graded sense) $R \odot A$-algebra whenever R is an A-algebra (recall that Sq is an algebra automorphism). If R is a polynomial of rank n, we define $\mathrm{Ad}_R = H_nR$. Now Ad_R is certainly a module over $R \odot A$, but we must explain why it is free of rank one as an R-module. In fact, when R is a polynomial, H_*R can be described very simply as follows: Let Ω_R^1 denote the module of 1-forms on R—i.e., the free R-module on symbols dx, $x \in R$, modulo the relation: $d(xy) = xdy + ydx$. Thus if $z_1, \ldots, z_n$ are polynomial generators for R, Ω_R^1 is a free R-module on $dz_1, \ldots, dz_n$. Furthermore, defining $\mathrm{Sq}(dx) = d\,\mathrm{Sq}\,x$, Ω_R^1 becomes a module over $R \odot A$. Hence the algebra Ω_R^* of differential forms becomes an $R \odot A$ algebra. Note: Ω_R^k is free of rank $\binom{n}{k}$ as an R-module.

PROPOSITION. *$\Omega_R^* \cong H_*R$ as $R \odot A$-algebras.*

PROOF. Define $\phi^1 \colon \Omega_R^1 \to H_1R$ by mapping adx to the class of $a \otimes x$ in the Hochschild complex (see Loday-Quillen [**L-Q**]). It is easy to see that this is an isomorphism of R-modules (for any commutative algebra R) and hence, in our context, an isomorphism of $R \odot A$-modules. Since H_*R is strictly anticommutative with respect to the Hochschild grading ($x \in H_kR$, k odd, then $x^2 = 0$), ϕ^1 extends to a homomorphism $\phi \colon \Omega_R^* \to H_*R$ of $R \odot A$-algebras. On the other hand, by using a Koszul resolution to compute H_*R, it is easy to see that ϕ is an isomorphism. (Indeed, this is a trivial special case of a theorem of Hochschild, Kostant, and Rosenberg [**H-K-R**].) Thus $H_nR = \Omega_R^n$ is a Thom module with "Thom class" $dz_1dz_2 \cdots dz_n$ (see also [**B-S**]).

PROOF OF THE THEOREM. (a) Let G_c denote G regarded as a left G-space via the conjugation action: $g \cdot x = gxg^{-1}$. Then it is easy to see that there is a pullback diagram up to homotopy

$$\begin{array}{ccc} EG \times_G G_c & \longrightarrow & BG \\ \downarrow & & \downarrow{\scriptstyle\Delta} \\ BG & \xrightarrow{\Delta} & BG \times BG \end{array} \tag{1}$$

where Δ is the diagonal map. (Thus, $EG \times_G G_c$ is homotopy equivalent to the free loop space on BG; see [**S2**].) Now if G is connected, so that $BG \times BG$ is simply connected, the Eilenberg-Moore spectral sequence associated to (1) converges to $H^*(EG \times_G G_c)$. Its E_2-term is precisely H_*R ($R = H^*BG$), and it is a spectral sequence of A-algebras. Furthermore, the spectral sequence collapses, since it is a second-quadrant cohomology spectral sequence and H_*R is generated by $H_0R = R$ and H_1R. Note that H_nR is therefore a quotient of $H^*(EG \times_G G_c)$. Now $\mathrm{Ad}_G = EG \times_G \mathrm{ad}_G$, where ad_G is the adjoint representation of G. Identifying ad_G with a G-invariant neighborhood U of the identity in G_c, we obtain a collapse map

$$EG \times_G G_c \xrightarrow{h} EG^+ \wedge_G (U/\partial U) \cong T(\mathrm{Ad}_G).$$

Let f denote the composite $H^*T(\mathrm{Ad}_G) \xrightarrow{h^*} H^*(EG \times_G G_c) \xrightarrow{\pi} H_nR$, where π is the projection. Then f is a map of $R \odot A$-modules, and it only remains to show that f maps Thom class to Thom class.

To see this, consider the commutative diagram

$$\begin{array}{ccccc} H^*T(\mathrm{Ad}_G) & \xrightarrow{h^*} & H^*EG \times_G G_c & \xrightarrow{\pi} & H_nR = R \cdot dz_1 \cdots dz_n \\ \downarrow & & \downarrow & & \downarrow \\ H^*S^{\dim G} & \longrightarrow & H^*G & \longrightarrow & \mathrm{Tor}_n^R(F_2, F_2) = F_2 \cdot dz_1 \cdots dz_n \end{array} \tag{2}$$

where the left-hand square arises from the diagram

$$\begin{tikzcd} T(\mathrm{Ad}_G) & EG \times_G G_c \arrow[l] \\ S^{\dim G} \arrow[u] & G \arrow[l] \arrow[u] \end{tikzcd}$$

and the right-hand one from the map of Eilenberg-Moore spectral sequences obtained by mapping the pullback diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & EG \arrow[d] \\ + \arrow[r] & BG \end{tikzcd} \tag{3}$$

into the diagram (1). Here S^k is the sphere of dimension k.

The spectral sequence of (3) also collapses. Furthermore, it is easy to see that the corresponding map $H_*R \to \mathrm{Tor}^R(F_2, F_2)$ coincides with the projection $H_*R \to (H_*R)/\overline{R}H_*R$, where $\overline{R}$ is the augmentation ideal. In particular, the map $H_nR \to \mathrm{Tor}_n^R(F_2F_2)$ is surjective. Since the bottom row of (2) is an isomorphism, it follows that πh^* is an isomorphism, as desired.

The case $G = O(n)$ is easy to check directly—see the example below.

(b) Let $R = F_2[z_1, \ldots, z_n]$, where $|z_i| = d_i$. Then $\mathrm{Sq}^{d_i}(dz_i) = d(z_i^2) = 0$. Hence $\mathrm{Sq}(dz_1 \cdots dz_n) = 0$ if $k > r$.

(c) This is a general property of Hochschild homology.

Now let $U = F_2[x_1, \ldots, x_n]$, $\dim x_i = 1$, with its unique unstable A-algebra structure. An A-algebra R will be called *abelian* if R is isomorphic to a subalgebra of U of the form $F_2[x_1^{2^{i}1}, \ldots, x_n^{2^{i}n}]$ for some nonnegative integers $i_1, \ldots, i_n$. If R is abelian, then clearly Ad_R is trivial. An embedding of polynomial A-algebras

$$R = F_2[z_1, \ldots, z_n] \hookrightarrow T = F_2[y_1, \ldots, y_n]$$

is *nonsingular* if the Jacobian determinant $J = \det[\partial z_i/\partial y_j]$ is nonzero.

(e) An embedding $R \xrightarrow{i} T$ is nonsingular if and only if the induced map $H_nR \xrightarrow{i_n} H_nT$ is nonzero, in which case i_* is an isomorphism onto $R \cdot (J dy_1 \cdots dy_n)$. But if T is abelian, this Thom module can be identified with $R \cdot J \subset T$ (in particular $R \cdot J$ is a sub-A-module of T).

Next, one knows from the work of Adams and Wilkerson [**A-W**] that any R of rank n can be embedded in U. Of course, the embedding can be singular, but on inspection, it appears that this can only happen for trivial reasons—i.e., $R \subset T$ for some proper abelian subalgebra T of U. This led us to conjecture that every R admits a nonsingular embedding in some abelian T (of the same rank, of course); this conjecture has been proven by Wilkerson [**W**]. This result is of interest in its own right, but here we simply note that part (d) of the theorem follows immediately: Choose a nonsingular embedding of R in an abelian T. Then $\mathrm{Ad}_R \cong R \cdot J$ as in (e), and if $J \neq 1$ (i.e., $R \neq T$) then A acts nontrivially on J. This completes the proof of the Theorem.

EXAMPLES. (a) Let $R = H^*BO(n) = U^{\Sigma_n} = F_2[w_1, \ldots, w_n]$. Then J is easily seen to be the Vandermonde determinant $\prod_{i<j}(x_i + x_j)$. Hence $\mathrm{Sq}\, J = (\prod_{i<j} 1 + x_i + x_j) \cdot J$ and $w(\mathrm{Ad}_R) = \prod_{i<j}(1 + x_i + x_j)$ (see [**B-H**, §15.4]).

(b) Let $R = U^{\mathrm{GL}_n F_2}$ be the Dickson algebra. Then $R = F_2[e_1, \dots, e_n]$, where $e_i = c_{n,n-i}$, and the A-action is given by

$$\mathrm{Sq}\, e_i = (e_i + \cdots + e_n)(1 + e_1 + \cdots + e_n) + \sum_{j>1} e_j^2.$$

Then $w(\mathrm{Ad}_R) = (1+e_1+\cdots+e_n)^{n-1}$. This follows either directly, or by computing $J = e_n^{n-1}$. For example, if $n = 3$ then $R \cong H^*(BG_2)$, and we have $w(\mathrm{Ad}_{G_2}) = 1 + e_1^2 + e_2^2 + e_3^2$. (The formula for $\mathrm{Sq}\, e_i$ can be derived using the methods of [**W**].)

REMARK. The most interesting case is when $R \subset U$ is already a nonsingular (i.e., separable) embedding. In that case, we see that $\dim \mathrm{Ad}_R$ is precisely r.

3. An alternative definition. Let $R = F_2[z_1, \dots, z_n]$ be a polynomial algebra which is an unstable algebra over A, and let $S = F_2[u_1, \dots, u_n]$ be a subalgebra invariant under A and having the same rank. Then $R \otimes_S F_2 = R/RS^+$, which we denote R/S, is a Poincaré duality algebra over the Steenrod algebra of dimension $N = \dim S - \dim R = \sum \dim u_i - \sum \dim z_i$. Let $\pi\colon R \to R/S$ be the quotient homomorphism. (See [**S1**].)

Since R/S is a Poincaré duality algebra, it has normal Stiefel-Whitney classes $\overline{w}_i(R/S) \in (R/S)^i$ characterized by $\overline{w}_i(R/S)x = (\mathrm{Sq}^{-1} x)^n$ for all $x \in (R/S)^{n-i}$. These are the Stiefel-Whitney classes corresponding to the contragredient module $(R/S)^{\#}$, called the normal Thom module.

PROPOSITION. *There are unique classes $\overline{w}_i \in R^i$ having the property that for every subalgebra $S \subset R$ of the same rank, $\pi\overline{w}_i = \overline{w}_i(R/S)$ is the normal Stiefel-Whitney class.*

PROOF. $R^{2^s} = F_2[z_1^{2^s}, \dots, z_n^{2^s}]$ is a subalgebra for which $\pi\colon R^i \to (R/R^{2^s})^i$ is an isomorphism if $i < 2^s$, and taking $2^s > i$ determines the class $\overline{w}_i$ by its projection. To see that this is well defined, we have inclusions

$$\begin{array}{ccc} S^{2^s} & \longrightarrow & S \\ \downarrow & & \downarrow \\ R^{2^s} & \longrightarrow & R \end{array}$$

for any subalgebra S, with $R^{2^s}/S^{2^s} \to R/S^{2^s} \xrightarrow{\pi_1} R/R^{2^s}$ and $S/S^{2^s} \to R/S^{2^s} \xrightarrow{\pi_2} R/S$ being Poincaré extensions in the sense of [**M**, Appendix B]. Then $\overline{w}_i(R/S^{2^s})$ and the projection of $\overline{w}_i$ in R/S^{2^s} have the same image under π_1 [**M**, Proposition B1], so coincide, since $2^s > i$; and then applying π_2 yields that $\overline{w}_i(R/S)$ is the projection of $\overline{w}_i$ into R/S, since it coincides with the projection of $\overline{w}_i(R/S^{2^s})$.

PROPOSITION. *In R, $\overline{w}_j = 0$ if $j > \dim R$.*

PROOF. According to Adams-Wilkerson [**A-W**], we have an inclusion $R \subset U = F_2[x_1, \dots, x_n]$ with $\dim x_i = 1$, where U is the algebraic closure of R. By [**M**, Theorem B3], $\overline{w}(U) = 1$. Using standard notation we let $[M]$ denote the fundamental homology class of the Poincaré algebra M. We may then choose a class $a \in U$ so that $\pi(a)[U/R] = 1$. Then in the sequence

$$R/R^{2^s} \xrightarrow{i} U/R^{2^s} \xrightarrow{\pi} U/R$$

we have $r[R/R^{2^s}] = air[U/R^{2^s}]$ and so

$$\begin{aligned}\overline{w}(R)r'[R/R^{2^s}] &= \mathrm{Sq}^{-1} r'[R/R^{2^s}] = ai\,\mathrm{Sq}^{-1} r'[U/R^{2^s}] \\ &= \mathrm{Sq}(ai\,\mathrm{Sq}^{-1} r')[U/R^{2^s}]\end{aligned}$$

since $\overline{w}(U) = 1$, and this is $\mathrm{Sq}\,a \cdot ir'[U/R^{2^s}]$. Since the largest nonzero term of $\mathrm{Sq}\,a$ is in dimension $2\dim(U/R)$, we have $\overline{w}(R)r'[R/R^{2^s}] = 0$ if $\dim r' < \dim(R/R^{2^s}) - \dim(U/R)$, or $\overline{w}_j(R) = 0$ if $j > \dim(U/R)$.

COROLLARY. *The classes $\overline{w}_i$ define a homomorphism $H^*BO(m) \to R$.*

PROOF. We clearly have a homomorphism $F_2[w_1, \dots, w_m] \to R$ with w_i mapping to $\overline{w}_i$. Projecting into the Poincaré algebra R/R^{2^s} for large s, the images are Stiefel-Whitney classes of a Poincaré algebra and satisfy all relations true in $H^*BO(m)$.

PROPOSITION. *If $R = H^*BH$ and $S \subset R$ is the subalgebra H^*BG for some compact Lie group G containing H, then the Stiefel-Whitney class of the adjoint bundle $w(\mathrm{Ad}_H)$ restricts to $\overline{w}(R/S)$ in R/S.*

PROOF. In the fibring $G/H \xrightarrow{i} BH \to BG$ it is well known that the bundle Ad_H restricts to the normal bundle of the manifold G/H. Thus $\pi(w(\mathrm{Ad}_H)) = i^*w(\mathrm{Ad}_H) \in R/S = H^*G/H$ is the Stiefel-Whitney class of the normal Thom module of R/S.

Of course, when $R = H^*BH$ there are not enough subalgebras $S \subset R$ of the form H^*BG with $H \subset G$ to make this lead to a proof that $\overline{w}(R) = w(\mathrm{Ad}_H)$. By extensive and unpleasant calculation we have verified this equality for all H known to have H^*BH polynomial. It is pointless to reproduce such calculations. Instead we are led to

PROBLEM. *Does the class $\overline{w} \in R$ defined by using subalgebras coincide with the class $w(\mathrm{Ad}_R)$ arising from Hochschild homology?*

REFERENCES

[**A-W**] J. F. Adams and C. W. Wilkerson, *Finite H-spaces and algebras over the Steenrod algebra*, Ann. of Math. (2) **111** (1980), 95–143.

[**B**] A. Borel, *Sous-groupes commutatifs et torsions des groupes de Lie compacts connexes*, Tôhoku Math. J. **13** (1961), 216–240.

[**B-H**] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces*. I, Amer. J. Math. **80** (1958), 458–538.

[**B-S**] P. Baum and L. Smith, *The real cohomology of differentiable fiber bundles*, Comment. Math. Helv. **42** (1967), 171–179.

[**C-E**] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N.J., 1956.

[**H**] D. Handel, *Thom modules*, J. Pure Appl. Algebra **36** (1985), 237–252.

[**H-K-R**] G. Hochschild, B. Kostant and A. Rosenberg, *Differential forms on regular affine algebras*, Trans. Amer. Math. Soc. **102** (1962), 383–408.

[**L-Q**] J. L. Loday and D. G. Quillen, *Cyclic homology and the Lie algebra homology of matrices*, Comment. Math. Helv. **59** (1984), 565–591.

[**M**] S. A. Mitchell, *Finite complexes with A(n)-free cohomology*, Topology **24** (1985), 227–246.

[**Q**] D. G. Quillen, *The spectrum of an equivariant cohomology ring*. I, II, Ann. of Math. (2) **94** (1971), 549–602.

[**S1**] L. Smith, *A note on the realization of graded complete intersection algebras by the cohomology of a space*, Quart. J. Math. Oxford. **33** (1982), 379–384.

[**S2**] ——, *On the characteristic zero cohomology of the free loop space*, Amer. J. Math. **103** (1981), 887–910.

[**W**] C. W. Wilkerson, *Inseparable forms of algebras over the Steenrod algebra*, preprint.

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