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MULTICOHERENCE OF SPACES OF THE FORM X/M

ALEJANDRO ILLANES M.

ABSTRACT. Let X be a connected, locally connected, normal T_1-space and let M be a closed connected, locally connected subspace of X. Suppose that X/M denotes the space obtained by identifying M in a single point, and that, for a connected space Y, $r(Y)$ denotes the multicoherence degree of Y. In this paper, we prove that if M is unicoherent, then $r(X) = r(X/M)$. As an application of this result we prove that if $X = A \cup B$, where A, B are closed subsets of X and $A \cap B$ is connected, locally connected and unicoherent, then $r(X) = r(A) + r(B)$. Also, we prove that if X/M is unicoherent, then $r(X) \leqslant r(M)$.

Introduction. Throughout this paper X will denote a connected, locally connected, normal T_1-space and M will denote a closed, connected, locally connected subspace of X. We will denote by X/M the space obtained by identifying M in a single point, and by β: $X \to X/M$ the natural identification.

If Y is any space, let $b_0(Y)$ denote the number of components of Y less than one (or ∞ if this number is finite). The *multicoherence degree*, $r(X)$, of X is defined by $r(X) = \sup\{b_0(A \cap B)$: A, B are closed connected subsets of X and $X = A \cup B\}$. If $r(X) = 0$, X is said to be *unicoherent.*

We will be interested in studying relations among $r(X)$, $r(M)$, and $r(X/M)$. An antecedent of this is the following theorem of R. F. Dickman, Jr. [**2**, Theorems 2.4 and 4.2]: If X is compact, M is unicoherent, and $X - M$ is connected, then X is unicoherent if and only if X/M is unicoherent. We will prove that if M is unicoherent, then $r(X) = r(X/M)$. Also we will prove that if X/M is unicoherent, then $r(X) \leqslant r(M)$. We will show, with an example, that the local connectivity of M is a necessary condition for these results. As a consequence of the equality $r(X) = r(X/M)$, when M is unicoherent, we will obtain that if $X = A \cup B$, where A, B are closed subsets of X such that $A \cap B$ is connected, locally connected and unicoherent, then $r(X) = r(A) + r(B)$.

To deduce the main theorems of this paper, we will use the equality $r(Y) = R(Y)$, where $R(Y)$ is the "the analytic multicoherence degree of Y" which was introduced by S. Eilenberg [**4**], who found that $r(Y) = R(Y)$ when Y is a compact, connected, locally connected metric space. Later, A. H. Stone [**10**] proved that this equality holds for all connected, locally connected, normal T_1-spaces. The definition of $R(Y)$ can be found also in [**11**].

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If f is a function, we will denote by $f\,|\,E$ the restriction of f to E. A *map* is a continuous function. A *region* of X is an open connected subset of X. We will denote by I the unit interval $[0,1]$, by $\mathbb{R}^2$ the Euclidean plane, and by $\mathbb{N}$ the set of positive integers. If $n \in \mathbb{N}$, we define $\bar{n} = \{1,2,\ldots,n\}$.

1. Multicoherence of spaces of the form X/M. Given $n \in \mathbb{N}$, we define $\mathscr{L}_n = \{(u,v) \in \mathbb{R}^2\colon (u-(2i-1))^2 + v^2 = 1$ for some $i \in \bar{n}\}$, $\mathscr{L}_n^+ = \{(u,v) \in \mathscr{L}_n\colon v \geqslant 0\}$, and $\mathscr{L}_n^- = \{(u,v) \in \mathscr{L}_n\colon v \leqslant 0\}$. We denote by $\mathscr{C}_n$ the universal covering space of $\mathscr{L}_n$ and by $\rho_n\colon \mathscr{C}_n \to \mathscr{L}_n$ the covering map. We identify $\mathscr{C}_1$ with the real line $\mathbb{R}$, $\mathscr{L}_1$ with the unitary circumference S, and $\rho = \rho_1$ with the map $\rho(t) = (\cos(t), \sin(t))$. If f is a map from a space Y in $\mathscr{L}_n$, we write $f \sim_n 1$ (or $f \sim 1$ if $n = 1$) provided there exists a map $g\colon Y \to \mathscr{C}_n$ such that $f = \rho_n \circ g$. For $i \in \bar{n}$, we define $\ell_i\colon \mathscr{L}_n \to S$ by

$$\ell_i(u,v) = \begin{cases} (u-(2i-1),v), & |u-(2i-1)| \leqslant 1, \\ (-1,0), & u \leqslant 2i-2, \\ (1,0), & u \geqslant 2i. \end{cases}$$

From Theorem 4 in [**6**] we have: If Y is a connected, locally connected, unicoherent space and $f\colon Y \to \mathscr{L}_n$ is a map, then $f \sim_n 1$.

1.1. LEMMA. *If $f\colon X \to \mathscr{L}_m$ is a map such that $f\,|\,M \sim_m 1$, then*:

(a) *There exists a region U of X such that $f\,|\,U \sim_m 1$.*

(b) *There exists a map $g\colon X \to \mathscr{L}_m$ which is homotopic to f and $g\,|\,M$ is constant.*

PROOF. (a) (compare with [**3**, (6), §2]). Since $\mathscr{C}_m$ is an ANR (normal), there exist an open subset V of X and a map $h\colon V \to \mathscr{C}_m$ such that $M \subset V$ and $\rho_m \circ (h\,|\,M) = f\,|\,M$. For each $x \in M$, we choose a region U_x of X such that

$$\text{diameter}(f(Ux) \cup \rho_m(h(Ux))) < \tfrac{1}{4}$$

and $x \in U_x \subset V$. Then there exists a map $g_x\colon U_x \to \mathscr{C}_m$ such that $\rho_m \circ g_x = f\,|\,U_x$ and $g_x(x) = h(x)$. Let $x, y \in M$ be such that $U_x \cap U_y \neq \varnothing$. Then there exists a map $k\colon f(U_x \cup U_y) \cup (\rho_m \circ h)(U_x \cup U_y) \to \mathscr{C}_m$ such that $\rho_m \circ k =$ identity and $k(f(x)) = h(x) = g_x(x)$. By the Unique Lifting Theorem, we have that $k \circ (f\,|\,U_x) = g_x$ and $k \circ \rho_m \circ (h\,|\,U_x \cup U_y) = h\,|\,U_x \cup U_y$. In particular, $k(f(y)) = h(y) = g_y(y)$, so that $k \circ (f\,|\,U_y) = g_y$. Thus $g_x\,|\,U_x \cap U_y = g_y\,|\,U_x \cap U_y$. Consider $U = \bigcup\{U_x\colon x \in M\}$ and let $g\colon U \to \mathscr{C}_m$ be the map which extends each g_x. Then $\rho_m \circ g = f\,|\,U$. Hence $f\,|\,U \sim_m 1$.

(b) Since $\mathscr{C}_m$ is contractible (see [**9**, Theorem 4.1, Chapter VI]), there exists a map $F\colon \mathscr{C}_m \times I \to \mathscr{C}_m$ and there exists a point $p \in \mathscr{C}_m$ such that $F(z,0) = p$ and $F(z,1) = z$ for each $z \in \mathscr{C}_m$. Let U, V be regions of X and let $h\colon U \to \mathscr{C}_m$ be a map such that $\rho_m \circ h = f\,|\,U$ and $M \subset V \subset \mathrm{Cl}_X(V) \subset U$. Suppose that $\sigma\colon X \to I$ is a map such that $\sigma(M) = 0$ and $\sigma(X - V) = 1$. Define $G\colon X \times I \to \mathscr{L}_m$ by

$$G(x,t) = \begin{cases} f(x) & \text{if } x \notin \mathrm{Cl}_X(V), \\ \rho_m(F(h(x),\, t + (1-t)\sigma(x))) & \text{if } x \in U; \end{cases}$$

and $g(x) = G(x,0)$.

Suppose that $X = A \cup B$, where A, B are closed connected subsets of X, and suppose that $\ell_0(A \cap B) \geqslant m$ ($m \in \mathbb{N}$). Let $D_1, \dots, D_{m+1}$ be closed pairwise disjoint, nonempty subsets of X such that $A \cap B = D_1 \cup \cdots \cup D_{m+1}$. Tietze's extension theorem implies that there exists a map $f\colon X \to \mathscr{L}_m$ such that $f(A) \subset \mathscr{L}_m^+$, $f(B) \subset \mathscr{L}_m^-$, and $f(D_{i+1}) = \{(2i, 0)\}$ for each $i \in \{0, 1, \dots, m\}$. For $i \in \overline{m}$, we define $f_i = \ell_i \circ f$.

1.2. LEMMA. *$f_1, \dots, f_m$ are linearly independent (that means that $f_1^{a_1} \cdot \cdots \cdot f_m^{a_m} \sim 1$, with $a_1, \dots, a_m$ integers, is possible only when $a_1 = \cdots = a_m = 0$).*

PROOF. We choose a point $x_0 \in D_{m+1}$. Let $a_1, \dots, a_m$ be integers such that $f_1^{a_1} \cdot \cdots \cdot f_m^{a_m} \sim 1$ and let $\varphi\colon X \to \mathbb{R}$ be a map such that $f_1^{a_1} \cdot \cdots \cdot f_m^{a_m} = \rho \circ \varphi$ and $\varphi(x_0) = 0$. For $i \in \overline{m}$, $f_i(A) \subset S^+$ and $f_i(B) \subset S^-$ so, we can define $f_i^+ = (\rho \mid [0, \pi])^{-1} \circ (f_i \mid A)$ and $f_i^- = (\rho \mid [-\pi, 0])^{-1} \circ (f_i \mid B)$. Then

$$\rho \circ (a_1 f_1^+ + \cdots + a_m f_m^+) = (\rho \circ f_1^+)^{a_1} \cdots (\rho \circ f_m^+)^{a_m} = \rho \circ (\varphi \mid A)$$

and

$$(a_1 f_1^+ + \cdots + a_m f_m^+)(x_0) = 0 = \varphi \mid A(x_0).$$

Since A is connected, we have that $a_1 f_1^+ + \cdots + a_m f_m^+ = \varphi \mid A$. Similarly, $a_1 f_1^- + \cdots + a_m f_m^- = \varphi \mid B$. Then, for all $i \in \overline{m}$ and $x_i \in D_i$,

$$\begin{aligned} 0 &= (a_1(f_1^+ - f_1^-) + \cdots + a_m(f_m^+ - f_m^-))(x_i) \\ &= a_1 0 + \cdots + a_{i-1} 0 + a_i 2\pi + \cdots + a_m 2\pi. \end{aligned}$$

Hence $a_1 = \cdots = a_m = 0$.

1.3. THEOREM. *If M is unicoherent, then $\imath(X) = \imath(X/M)$.*

PROOF. It is easy to prove that $\imath(X/M) \leqslant \imath(X)$ ($\beta^{-1}(D)$ is connected for any closed connected subset D of X/M). Suppose that $m \in \mathbb{N}$ is such that $\imath(X/M) < m \leqslant \imath(X)$. Let $X = A \cup B$, where A and B are connected closed sets with $\ell_0(A \cap B) \geqslant m$, and let $f\colon X \to \mathscr{L}_m$ and $f_1, \dots, f_m\colon X \to S$ be as in Lemma 1.2.

Since M is unicoherent, $f \mid M \sim_m 1$. By Lemma 1.1, there exists a map $g\colon X \to \mathscr{L}_m$ such that $g \mid M$ is constant and g is homotopic to f. Let $k\colon X/M \to \mathscr{L}_m$ be a map such that $g = k \circ \beta$. Define $X^+ = k^{-1}(\mathscr{L}_m^+)$, $X^- = k^{-1}(\mathscr{L}_m^-)$, and $k_1 = \ell_1 \circ k, \dots, k_m = \ell_m \circ k$. Then $X/M = X^+ \cup X^-$; X^+, X^- are closed subsets of X/M, and $k_i \mid X^+ \sim 1$, $k_i \mid X^- \sim 1$ for each $i \in \overline{m}$. Since $\imath(X/M) < m$, by [**10**, Theorem 5], there exists integers $a_1, \dots, a_m$ not all zero such that $k_1^{a_1} \cdot \cdots \cdot k_m^{a_m} \sim 1$. But $f_1^{a_1} \cdot \cdots \cdot f_m^{a_m}$ is homotopic to $(k_1^{a_1} \cdot \cdots \cdot k_m^{a_m}) \circ \beta$, so (see [**8**, Lemma 5]) $f_1^{a_1} \cdot \cdots \cdot f_m^{a_m} \sim 1$. This contradiction to Lemma 1.2 completes the proof.

REMARKS. Notice that in Theorem 1.3 we only need M to be a closed connected subset of X such that for any $m \in \mathbb{N}$ and any map $f\colon X \to \mathscr{L}_m$, $f \mid M \sim_m 1$. In [7], it was proved that if F is a subset of X with n elements ($n \in \mathbb{N}$), then $\imath(X/F) = \imath(X) + n - 1$. From here we can state the following

1.4. COROLLARY. *Let N be a closed subset of X with n components such that for any $m \in \mathbb{N}$ and any map $f\colon X \to \mathscr{L}_m$, $f \mid N \sim_m 1$. (This is true if each component of N is locally connected and unicoherent.) Then $\imath(X/N) = \imath(X) + n - 1$.*

1.5. THEOREM. *If X/M is unicoherent, then $\imath(X) \leqslant \imath(M)$.*

PROOF. Suppose that $\imath(X) \geqslant m > \imath(M)$ ($m \in \mathbb{N}$). Let A, B, f: $X \to \mathscr{L}_m$, and $f_1, \ldots, f_m$: $X \to S$ be as in Lemma 1.2. Since $f_i \mid A \cap M \sim 1$ and $f_i \mid B \cap M \sim 1$ for any $i \in \overline{m}$, we have, by Theorem 5 of [**10**], that there exist integers $a_1, \ldots, a_m$ not all zero such that $(f_1^{a_1} \cdot \cdots \cdot f_m^{a_m}) \mid M \sim 1$. Then there exists a map g: $X \to S$ homotopic to $f_1^{a_1} \cdot \cdots \cdot f_m^{a_m}$ such that $g \mid M$ is constant. Let k: $X/M \to S$ be a map such that $g = k \circ \beta$. Since X/M is unicoherent, we have that $k \sim 1$. This implies that $g \sim 1$ and, consequently, $f_1^{a_1} \cdot \cdots \cdot f_m^{a_m} \sim 1$. This contradiction ends the proof.

REMARK. Theorems 1.3 and 1.5 suggest the possibility that $\imath(X) \leqslant \imath(X/M) + \imath(M)$ always holds. This is not true as is shown by the following

1.6. EXAMPLE. Let X be a torus of genus two with an open disk removed. Let M be the boundary of X. Then M is homeomorphic to S and X/M is homeomorphic to the torus of genus two. It is easy to prove that there exists a subspace of X homeomorphic to $\mathscr{L}$, which is a deformation retract of X, so that $\imath(X) = 4$ [**4**, §2, Theorem 4]. On the other hand, $\imath(X/M) = 2$ (see [**5**]) and $\imath(M) = 1$.

1.7. COROLLARY. *Suppose that Z is a locally connected T_1-compactification of a connected, locally compact space Y. Suppose also that $Z - Y$ is connected and locally connected. We denote by Y_∞ the one-point compactification of Y. Then*:

(a) *If $Z - Y$ is unicoherent, $\imath(Z) = \imath(Y_\infty)$.*

(b) *If Y_∞ is unicoherent, $\imath(Z) \leqslant \imath(Z - Y)$.*

REMARKS. In [**7**], some ways of calculating $\imath(Y_\infty)$ for a connected, locally connected, locally compact T_1-space are given. The local connectivity of M is a necessary condition in Theorems 1.3 and 1.5, and in Corollary 1.7 as is shown by the following example.

1.8. EXAMPLE. Let $X = \{(u, v) \in \mathbb{R}^2 \colon 1 \leqslant u^2 + v^2 \leqslant 4\}$ and let

$$M = \left\{((1 + 2\exp(t))/(1 + \exp(t)))(\cos(t), \sin(t)) \in \mathbb{R}^2 \colon t \in \mathbb{R}\right\} \cup \left\{(u, v) \in \mathbb{R}^2 \colon u^2 + v^2 = 1 \text{ or } u^2 + v^2 = 4\right\}.$$

Then M is a closed connected subset of X. It is easy to prove that M is unicoherent, $X - M$ is homeomorphic to $\mathbb{R}^2$, and X is a compactification of $\mathbb{R}^3$. Then X/M is homeomorphic to a 2-sphere. Hence $\imath(X/M) = 0 = \imath(M)$, while $\imath(X) = 1$.

We end this section showing a case in which X/N is unicoherent.

1.9. THEOREM. *Suppose that N is a closed connected subset of X and that there exists a map g: $X \to X$ such that $g(X) \subset N$ and g is homotopic to the identity of X. Then X/N is unicoherent.*

PROOF. Since X/N is normal, by Lemma 1.2, it is enough to prove that if f: $X/N \to S$ is a map, then $f \sim 1$. From the hypothesis, we have that $f \circ h \sim 1$ where h: $X \to X/N$ is the identification map. Let k: $X \to \mathbb{R}$ be a map such that $\rho \circ k = f \circ h$. Then $k \mid N$ is constant, so there exists a map l: $X/N \to \mathbb{R}$ such that $l \circ h = k$. Then $\rho \circ l \circ h = f \circ h$, so that $\rho \circ l = f$. Therefore, X/N is unicoherent.

1.10. COROLLARY. *If N is a deformation retract of X, then X/N is unicoherent.*

2. Multicoherence of sums. Throughout this section, A and B will denote closed, nonempty subsets of X such that $A \cap B$ is connected and $X = A \cup B$. As a special case of Theorem 7 in **[10]** we have that if A, B are locally connected, then $\imath(X) \leqslant \imath(A) + \imath(B)$. Theorem 2.1 gives sufficient conditions in order that the equality $\imath(X) = \imath(A) + \imath(B)$ holds. This is a generalization of Corollary 7 in **[1]** which says that if X is compact, $A \cap B$ and X are unicoherent, and $A \cap B$ is locally connected, then A and B are unicoherent. We will use the following result which was proved in [7]: If p is any point of X and $\mathscr{D}$ is the family of components of $X - \{p\}$, then $\imath(X) = \Sigma_{D \in \mathscr{D}} \imath(S \cup \{p\})$.

2.1. THEOREM. *If $A \cap B$ is locally connected and unicoherent, then $\imath(X) = \imath(A) + \imath(B)$.*

PROOF. Let $M = A \cap B$. By Theorem 1.3, $\imath(X) = \imath(X/M)$, $\imath(A) = \imath(A/M) = \imath(\beta(A))$, and $\imath(B) = \imath(B/M) = \imath(\beta(B))$ (β: $X \to X/M$ is the identification map). We put $\{p\} = \beta(M)$, $\mathscr{D} = \{D$: D is component of $X/M - \{p\}\}$, $\mathscr{D}_A = \{D \in \mathscr{D}$: $D \subset \beta(A)\} = \{D$: D is component of $\beta(A) - \{p\}\}$, and $\mathscr{D}_B = \{D \in \mathscr{D}$: $D \subset \beta(B)\} = \{D$: D is component of $\beta(B) - \{p\}\}$. Then

$$\imath(X/M) = \sum_{D \in \mathscr{D}} \imath(D \cup \{p\}) = \sum_{D \in \mathscr{D}_A} \imath(D \cup \{p\}) + \sum_{D \in \mathscr{D}_B} \imath(D \cup \{p\})$$
$$= \imath(\beta(A)) + \imath(\beta(B)).$$

Hence, $\imath(X) = \imath(A) + \imath(B)$.

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INSTITUTO DE MATEMÁTICAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MEXICO, CIUDAD UNIVERSITARIA, D.F. C.P. 04510, MEXICO