KNOTS WITH FINITE WEIGHT COMMUTATOR SUBGROUPS

CHARLES LIVINGSTON

Abstract. An example of a knot in $S^3$ is constructed which has a companion of winding number zero but for which the commutator subgroup of the fundamental group of the complement is of finite weight. This provides a counterexample to a conjecture made by Jonathan Simon.

A conjecture made by J. Simon states: If a knot $K \subset S^3$ has a companion of winding number zero, then the commutator subgroup of $\pi_1(S^3 - K)$ is of infinite weight. The conjecture appears in Problem 1.14 of Kirby's problem list [K]. This paper presents a counterexample. The knot $J \subset S^1 \times B^2$ illustrated in Figure 1 has the property that for any embedding $f: S^1 \times B^2 \to S^3$, the commutator subgroup of $\pi_1(S^3 - f(J))$ is of finite weight. (In Figure 1, $S^1 \times B^2$ is viewed as the complement in $\mathbb{R}^3$ of the $z$-axis, indicated by the $\otimes$, in a projection onto the $x$-$y$ plane.)

Figure 1

Preliminaries. A reference for the results and techniques of knot theory used here is Rolfsen's book [R].

A knot $K \subset S^3$ has a companion if there is an embedding $f$ of $S^1 \times B^2$ into $S^3$ with $K \subset f(S^1 \times B^2)$ and with $f(S^1 \times \partial B^2)$ incompressible in $S^3 - K$. In this case $f(S^1 \times \{0\})$ is called a companion of $K$. The winding number of the companion is the homology class in $H_1(f(S^1 \times B^2))$ represented by $K$.

The weight of a group is the minimum number of elements which can normally generate it.

Received by the editors July 9, 1986.
1980 Mathematics Subject Classification (1985 Revision). Primary 57M25.
Key words and phrases. Knot theory, companion, commutator subgroup.
Supported in part by a grant from the NSF.
To prove that the commutator subgroup of a knot group has finite weight, we will consider the following situation. \( F \) will denote an oriented Seifert surface for a knot \( K \subset S^3 \). The positive and negative pushoffs of \( F \) into \( S^3 - F \) will be denoted by \( i_+ \) and \( i_- \).

**Proposition.** If both \( \pi_1(S^3 - F)/\langle i_+(\pi_1(F)) \rangle \) and \( \pi_1(S^3 - F)/\langle i_-(\pi_1(F)) \rangle \) are trivial, then the commutator subgroup of \( \pi_1(S^3 - K) \) is of finite weight.

**Proof.** The commutator subgroup of \( \pi_1(S^3 - K) \) is isomorphic to \( \pi_1(S^3 - K) \), where \( S^3 - K \) denotes the infinite cyclic cover of \( S^3 - K \). Let \( N(F) \) denote an open regular neighborhood of \( F \). Then \( S^3 - K \) is homeomorphic to the infinite union

\[
\cdots \cup (S^3 - N(F)) \cup (S^3 - N(F)) \cup (S^3 - N(F)) \cup \cdots
\]

where \( i_-(F) \) in each copy of \( S^3 - N(F) \) is identified with \( i_+(F) \) in the next copy.

This decomposition of \( S^3 - K \) induces a decomposition of \( \pi_1(S^3 - K) \) as

\[
\cdots \g_2 \ast \g_1 \ast \g_0 \ast \g_1 \ast \g_2 \ast \cdots
\]

with each \( \g_i \equiv \pi_1(S^3 - F) \) and \( H_i \equiv \pi_1(F) \). We are not assuming that \( i_+ \) or \( i_- \) induce injections on \( H_i \).

Any element in \( G_n \) is a product of conjugates of elements in \( i_+(H_{n-1}) \), since \( G_n/\langle i_+(H_{n-1}) \rangle = 1 \). Hence any element in \( G_n \) is the product of conjugates of elements in \( i_-(H_{n-1}) \subset G_{n-1} \). Similarly, any element in \( G_n \) is the product of conjugates of elements in \( G_{n+1} \). Proceeding by induction, any element in \( \pi_1(S^3 - K) \) is the product of conjugates of elements in \( G_0 \). Since \( \pi_1(S^3 - F) = G_0 \) is finitely generated, \( \pi_1(S^3 - K) \) is of finite weight.

\( \pi_1(S^3 - f(J)) \) has finite weight commutator subgroup. Figure 2 provides a second illustration of \( J \subset S^1 \times B^2 \). A Seifert surface \( F \) consists of four bands joined as indicated in the figure. \( F \) is oriented so that at \( \alpha \cap \beta \) the positive normal points at

![Figure 2](https://www.ams.org/journal-terms-of-use)
the viewer. A set of normal generators for \( \pi_1(S^1 \times B^2 - F) \), \( \{ 1, x, y, z, w \} \), is indicated, as is a set of loops on \( F \), \( \{ \alpha, \beta, \gamma, \delta \} \).

Let \( f \) be an embedding of \( S^1 \times B^2 \) into \( S^3 \) with \( f(S^1 \times \{0\}) \) representing a nontrivial knot \( K \). The torus \( f(S^1 \times \partial B^2) \), which we will later see is incompressible in \( S^3 - f(J) \), induces a decomposition

\[
\pi_1(S^3 - f(F)) \cong \pi_1(S^1 \times B^2 - F) \ast_{\pi_1(S^3 - K)} \pi_1(S^1 \times \partial B^2).
\]

Denote \( i_+(\pi_1(f(F))) \) by \( N_+ \). To apply the proposition we need to show that \( \pi_1(S^3 - f(F))/\langle N_+ \rangle \) is trivial.

We observe first that the meridian to \( K \) is trivial in this quotient group, as it is represented by \( i_+(\beta) \). Since the meridian to \( K \) normally generates \( \pi_1(S^3 - K) \), we have \( \pi_1(S^3 - K) \subset \langle N_+ \rangle \), and in particular, \( l \in \langle N_+ \rangle \).

It now suffices to show that \( \pi_1(S^1 \times B^2 - F)/\langle i_+(\pi_1(F)), l \rangle \) is trivial. By considering \( i_+(\alpha), i_+(\beta), i_+(\gamma), \) and \( i_+(\delta) \) we see that in addition to \( l \), the four elements \( lzw, yx, wlz, \) and \( y \) are trivial in this quotient group. It follows immediately that \( l, x, y, z, \) and \( w \) are all trivial in the quotient. As these normally generate \( \pi_1(S^1 \times B^2 - F) \), the quotient group is trivial.

A similar argument applies for \( i_-(\pi_1(f(F))) \). In this case the meridian to \( K \) is represented by \( i_-(\delta) \). Consideration of \( i_-(\alpha), i_-(\beta), i_-(\gamma), \) and \( i_-(\delta) \) shows that in addition to \( l \), each of \( lzw, x, wl, \) and \( yx \) are trivial in

\[
\pi_1(S^1 \times B^2 - F)/\langle i_-(\pi_1(F)), l \rangle.
\]

Again it follows that the quotient is trivial.

\( f(S^1 \times \partial B^2) \) is incompressible in \( S^3 - f(J) \). To prove that \( f(J) \) has a companion it is necessary to show that \( f(S^1 \times \partial B^2) \) is an incompressible torus in \( S^3 - f(J) \). As the knot \( K \) is nontrivial, it suffices to show that \( S^1 \times \partial B^2 \) is incompressible in \( S^1 \times B^2 - J \).

There are disjoint properly embedded discs \( D_1 \) and \( D_2 \) in \( S^1 \times B^2 \) such that cutting \( (S^1 \times B^2, J) \) along \( D_1 \) and \( D_2 \) results in tangles \( T_1 \) and \( T_2 \) (illustrated in Figure 3).

![Figure 3](https://www.ams.org/journal-terms-of-use)
Suppose there is a properly embedded disc $D$ in $S^1 \times B^2$ disjoint from $J$, and with $\partial D$ a meridian to $S^1 \times B^2$. It follows from a standard innermost circle argument that there is such a $D$ with $D \cap (D_1 \cup D_2) = \emptyset$. (Sketch of proof: First arrange that $\partial D \cap (D_1 \cup D_2) = \emptyset$ via an isotopy. If $D \cap (D_1 \cup D_2)$ contains any trivial circles on $(D_1 \cup D_2) - J$, swap a disc on $D$ with an innermost such disc on $(D_1 \cup D_2)$. If after repeating this procedure to eliminate all such circles of intersection any circles of intersection remain, an innermost disc on $D$ along with an annulus from one of the $D_i$ will form a new disc $D'$ with the desired properties.)

Hence, if $S^1 \times \partial B^2$ is compressible in $S^1 \times (B^2 - J)$, the horizontal equator bounds an embedded disc in either $B^3 - T_1$ or $B^3 - T_2$. This is impossible for $T_1$ by linking number arguments. Kirby and Lickorish prove it is impossible for $T_2$ in [KL].

REFERENCES


Department of Mathematics, Indiana University, Bloomington, Indiana 47405