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CHEBYSHEV TYPE ESTIMATES FOR BEURLING GENERALIZED PRIME NUMBERS

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ABSTRACT. We consider a Beurling generalized prime system for which the distribution function $N(x)$ of the integers satisfies

$$\int_1^\infty x^{-1}\left\{\sup_{x\leqslant y}\frac{|N(y)-Ay|}{y}\right\}dx<\infty$$

with constant $A>0$. We shall prove that the Chebyshev type estimates

$$0<\liminf_{x\to\infty}\frac{\psi(x)}{x},\qquad \limsup_{x\to\infty}\frac{\psi(x)}{x}<\infty$$

hold for the system. This gives a partial proof of one of Diamond's conjectures.

Chebyshev was the first to establish the correct order of magnitude of the weighted counting function $\psi(x)$ of the ordinary prime numbers. He showed that there exist two numbers $\alpha>0$ and $\beta<\infty$ such that

$$\liminf_{x\to\infty}\frac{\psi(x)}{x}\geqslant\alpha,\qquad \limsup_{x\to\infty}\frac{\psi(x)}{x}\leqslant\beta. \tag{1}$$

The prime number theorem (P.N.T.) asserts that $\alpha=\beta=1$. Here we shall study Chebyshev type estimates for Beurling generalized primes.

Let $\mathscr{P}=\{p_i\}_{i=1}^\infty$, where $1<p_1\leqslant p_2\leqslant\cdots$, $p_i\to\infty$, be a set of Beurling generalized (henceforth g-) prime numbers and $\mathscr{N}=\{n_i\}_{i=0}^\infty$ be the associated set of g-integers (see [**1, 2**]). Define

$$N(x)=\sum_{\substack{i\\ n_i\leqslant x}}1,\qquad \psi(x)=\sum_{\substack{i,\alpha\\ p_i^\alpha\leqslant x}}\log p_i.$$

Beurling [**2**] proved that if

$$N(x)=Ax+O(x\log^{-\gamma}x) \tag{2}$$

for some constants $A>0$ and $\gamma>3/2$, then the P.N.T. holds for $\mathscr{P}$. If $\gamma=3/2$ in (2), the P.N.T. need not hold as Diamond [**3**] showed by an example based on a continuous example of Beurling. Diamond [**4**] also showed that if $\gamma>1$ in (2) then (1) holds. On the other hand, (1) is not generally true if $\gamma<1$ in (2), as an example

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of Hall [6] shows. Moreover, Diamond [5] conjectured that the weaker condition

$$\int_1^\infty x^{-2}|N(x) - Ax|\,dx < \infty$$

is sufficient to imply Chebyshev type bounds for a g-prime system.

In the present paper, we shall prove the following

THEOREM. *If the distribution function $N(x)$ of the g-integers of a g-prime system satisfies*

$$\int_1^\infty x^{-1}\left\{\sup_{x \leqslant y} \frac{|N(y) - Ay|}{y}\right\} dx < \infty \tag{3}$$

with constant $A > 0$, then there exist numbers $\alpha > 0$ and $\beta < \infty$ for which (1) *holds.*

This theorem gives a partial proof of Diamond's conjecture.

We divide the proof of the Theorem into several lemmas.

LEMMA 1. *Assume* (3). *Then there exists a function $Q(x)$ such that*

$$Q(x) \text{ is nonincreasing,} \tag{4}$$

$$\int_1^\infty Q(x)x^{-1}\,dx < \infty, \tag{5}$$

$$Q(x) \leqslant 4Q(x^2) \quad \text{for all } x \geqslant 1, \tag{6}$$

and

$$Q(x) \geqslant \sup_{x \leqslant y} \frac{|N(y) - Ay|}{y}. \tag{7}$$

PROOF. Let

$$Q_1(x) = \sup_{x \leqslant y} \frac{|N(y) - Ay|}{y}.$$

We note that $Q_1(x)$ is nonnegative and nonincreasing. Define $Q(x)$ recursively by setting

$$Q(x) = \begin{cases} Q_1(1) & \text{for } 1 \leqslant x \leqslant 2, \\ \max\{Q_1(2^{2^{m-1}}), 4^{-1}Q(2^{2^{m-1}})\} & \text{for } 2^{2^{m-1}} < x \leqslant 2^{2^m},\ m \in \mathbf{N}. \end{cases}$$

We now verify that this function satisfies each of the following conditions:

(i) $Q(x) \geqslant Q_1(x)$ (this is obvious).

(ii) $Q(x)\downarrow$.

We note that $Q(x)$ is constant on $2^{2^{m-1}} < x \leqslant 2^{2^m}$. If $1 \leqslant x_1 < 2 < x_2 \leqslant 2^2$, then

$$Q(x_2) = \max\{Q_1(2), 4^{-1}Q(2)\} \leqslant Q_1(1) = Q(x_1).$$

If $2^{2^{m-1}} < x_1 \leqslant 2^{2^m} < x_2 \leqslant 2^{2^{m+1}}$, then

$$Q(x_2) = \max\{Q_1(2^{2^m}), 4^{-1}Q(2^{2^m})\} \leqslant Q(2^{2^m}) = Q(x_1)$$

since, by (i), $Q_1(2^{2^m}) \leqslant Q(2^{2^m})$.

(iii) $Q(x) \leqslant 4Q(x^2)$.

For $1 \leqslant x \leqslant 2$, we have $1 \leqslant x^2 \leqslant 2^2$. If $1 \leqslant x^2 \leqslant 2$, we have nothing to show. If $2 < x^2 \leqslant 2^2$, we have

$$Q(x^2) \geqslant 4^{-1}Q(2) = 4^{-1}Q(x).$$

For $2^{2^{m-1}} < x \leqslant 2^{2^m}$, $m \geqslant 1$, we have $2^{2^m} < x^2 \leqslant 2^{2^{m+1}}$ and hence

$$Q(x^2) \geqslant 4^{-1}Q(2^{2^m}) = 4^{-1}Q(x).$$

(iv) $\int_1^\infty Q(x)x^{-1}\,dx < \infty$.

We have to show that

$$\sum_{m=1}^{\infty} \int_{2^{2^{m-1}}}^{2^{2^m}} Q(x)x^{-1}\,dx < \infty.$$

There are two distinct cases that we have to consider separately.

Case I. $4^{-1}Q(2^{2^m}) \geqslant Q_1(2^{2^m})$ for all $m \geqslant m_0$. In this case, for $2^{2^{m_0}} < x \leqslant 2^{2^{m_0+1}}$,

$$Q(x) = \max\left\{Q_1(2^{2^{m_0}}), 4^{-1}Q(2^{2^{m_0}})\right\} = 4^{-1}Q(2^{2^{m_0}}).$$

By induction, for all $m \geqslant m_0 + 1$, $2^{2^{m-1}} < x \leqslant 2^{2^m}$, we have

$$Q(x) = 4^{-1}Q(2^{2^m}) = 4^{-(m-m_0)}Q(2^{2^{m_0}}).$$

Therefore,

$$\sum_{m=m_0+1}^{\infty} \int_{2^{2^{m-1}}}^{2^{2^m}} Q(x)x^{-1}\,dx = \sum_{m=m_0+1}^{\infty} 4^{-(m-m_0)}Q(2^{2^{m_0}}) \int_{2^{2^{m-1}}}^{2^{2^m}} x^{-1}\,dx$$

$$= \sum_{m=m_0+1}^{\infty} 4^{-(m-m_0)}Q(2^{2^{m_0}}) \log(2^{2^{m-1}}) = Q(2^{2^{m_0}}) \log(2^{2^{m_0-1}}).$$

Case II. There exist $m_1 < m_2 < \cdots < m_k < m_{k+1} < \cdots$ such that $Q_1(2^{2^{m_k}}) > 4^{-1}Q(2^{2^{m_k}})$ and $Q_1(2^{2^m}) \leqslant 4^{-1}Q(2^{2^m})$ $\forall m \neq m_k$. In this case, we can show that

$$\sum_{m=m_k+1}^{m_{k+1}} \int_{2^{2^{m-1}}}^{2^{2^m}} Q(x)x^{-1}\,dx \leqslant 4Q_1(2^{2^{m_k}}) \log\left(2^{2^{m_k-1}}\right).$$

Actually, if $m_{k+1} = m_k + 1$, the left-hand side equals

$$\int_{2^{2^{m_k}}}^{2^{2^{m_k+1}}} Q(x)x^{-1}\,dx = Q_1(2^{2^{m_k}}) \int_{2^{2^{m_k}}}^{2^{2^{m_k+1}}} x^{-1}\,dx$$

$$= Q_1(2^{2^{m_k}}) \log(2^{2^{m_k}}) = 2Q_1(2^{2^{m_k}}) \log\left(2^{2^{m_k-1}}\right).$$

Therefore, we consider $m_k < m_k + 2 \leqslant m_{k+1}$. For $2^{2^{m_k}} < x \leqslant 2^{2^{m_k+1}}$, we have

$$Q(x) = \max\left\{Q_1(2^{2^{m_k}}), 4^{-1}Q(2^{2^{m_k}})\right\} = Q_1(2^{2^{m_k}}).$$

Then, for $2^{2^{m_k+1}} < x \leqslant 2^{2^{m_k+2}}$,

$$Q(x) = 4^{-1}Q\left(2^{2^{m_k+1}}\right) = 4^{-1}Q_1(2^{2^{m_k}}).$$

By induction, for all $m \in [m_k + 1, m_{k+1}]$ and $2^{2^{m-1}} < x \leqslant 2^{2^m}$, we have $Q(x) = 4^{-(m-m_k)+1}Q_1(2^{2^{m_k}})$. Therefore,

$$\begin{aligned}\sum_{m=m_k+1}^{m_{k+1}} \int_{2^{2^{m-1}}}^{2^{2^m}} Q(x)x^{-1}\,dx &= \sum_{m=m_k+1}^{m_{k+1}} 4^{-(m-m_k)+1}Q_1(2^{2^{m_k}}) \int_{2^{2^{m-1}}}^{2^{2^m}} x^{-1}\,dx \\ &= \sum_{m=m_k+1}^{m_{k+1}} 4^{-(m-m_k)+1}Q_1(2^{2^{m_k}}) \log(2^{2^{m-1}}) \\ &= Q_1(2^{2^{m_k}}) \log(2^{2^{m_k-1}}) 4 \sum_{k=1}^{m_{k+1}-m_k} 2^{-k} \leqslant 4Q_1(2^{2^{m_k}}) \log(2^{2^{m_k-1}}).\end{aligned}$$

Thus we have

$$\begin{aligned}\sum_{m=m_1+1}^{\infty} \int_{2^{2^{m-1}}}^{2^{2^m}} Q(x)x^{-1}\,dx &= \sum_{k=1}^{\infty} \sum_{m=m_k+1}^{m_{k+1}} \int_{2^{2^{m-1}}}^{2^{2^m}} Q(x)x^{-1}\,dx \\ &\leqslant 4 \sum_{k=1}^{\infty} Q_1(2^{2^{m_k}}) \log(2^{2^{m_k-1}}) \leqslant 4 \sum_{m=1}^{\infty} Q_1(2^{2^m}) \log(2^{2^{m-1}}).\end{aligned}$$

To show the last sum is finite, we note that

$$\int_{2^{2^{m-1}}}^{2^{2^m}} Q_1(x)x^{-1}\,dx \geqslant Q_1(2^{2^m}) \int_{2^{2^{m-1}}}^{2^{2^m}} x^{-1}\,dx = Q_1(2^{2^m}) \log(2^{2^{m-1}})$$

since $Q_1(x)\downarrow$. Therefore, we have

$$\sum_{m=1}^{\infty} Q_1(2^{2^m}) \log(2^{2^{m-1}}) \leqslant \sum_{m=1}^{\infty} \int_{2^{2^{m-1}}}^{2^{2^m}} Q_1(x)x^{-1}\,dx = \int_2^{\infty} Q_1(x)x^{-1}\,dx < \infty$$

by (3). This completes the proof of Lemma 1. □

LEMMA 2. *Assume* (4) *and* (5). *Then* $Q(x) = o(\log^{-1} ex)$.

PROOF. Given $\varepsilon > 0$, for $x \geqslant x_0$, we have $\int_x^\infty Q(t)t^{-1}\,dt < \varepsilon/2$. Thus, for $x \geqslant x_0^2$,

$$Q(x) \log\sqrt{x} \leqslant \int_{\sqrt{x}}^{x} Q(t)t^{-1}\,dt < \varepsilon/2,$$

i.e., $Q(x)\log x < \varepsilon$. □

The following lemma shows that $Q(x)\,dx$ has a kind of "stability" under multiplicative convolution.

LEMMA 3. *Assume* (4), (5), *and* (6). *Then* $\int_1^x t^{-1}Q(x/t)Q(t)\,dt \leqslant c_1 Q(x)$, *where* c_1 *is a constant.*

PROOF. We have

$$\begin{aligned}\int_1^x t^{-1}Q(x/t)Q(t)\,dt &= 2\int_1^{\sqrt{x}} t^{-1}Q(x/t)Q(t)\,dt \\ &\leqslant 2Q(\sqrt{x}) \int_1^{\sqrt{x}} Q(t)t^{-1}\,dt \leqslant c_1 Q(x)\end{aligned}$$

since, by (4) and (6), $Q(x/t) \leqslant Q(\sqrt{x}) \leqslant 4Q(x)$ for $1 \leqslant t \leqslant \sqrt{x}$. □

By using the "stability" of $Q(x)\,dx$, we can prove the following lemma which is the main step in the proof of the Theorem.

LEMMA 4. *Assume* (4), (5), (6), *and* (7). *Then, for fixed and sufficiently small* $\varepsilon > 0$, *we have*

$$u_\varepsilon(x) := \int_1^x dN*(\delta - \varepsilon t^{-\varepsilon}dt)*Q(t)\,dt \geqslant 0$$

for $x \geqslant 1$ *and* $u_\varepsilon(x) \to \infty$ *as* $x \to \infty$.

PROOF. If $1 \leqslant x \leqslant 1/\varepsilon$, then

$$\int_1^x dN*(\delta - \varepsilon t^{-\varepsilon}dt) \geqslant N(x) - \varepsilon N(x)(x-1) \geqslant 0.$$

The lemma is certainly true for $1 \leqslant x \leqslant 1/\varepsilon$, since the third convolution factor is everywhere nonnegative.

If $x > 1/\varepsilon$, we utilize all the convolution factors. We write

$$\begin{aligned} u_\varepsilon(x) &= A\int_1^x (\delta + dt)*(\delta - \varepsilon t^{-\varepsilon}dt)*Q(t)\,dt \\ &\quad + \int_1^x (dN - A\delta - A\,dt)*(\delta - \varepsilon t^{-\varepsilon}dt)*Q(t)\,dt \\ &= I_1 + I_2, \end{aligned}$$

say. We will show that I_1 is positive, that $I_1 \to \infty$ as $x \to \infty$ and that I_2 is negligible. Actually, we have

$$I_1 = A\int_1^x (x/t)^{1-\varepsilon}Q(t)\,dt = Ax\int_1^x u^{-\varepsilon-1}Q(x/u)\,du$$

since $\int_1^x(\delta + dt)*(\delta - \varepsilon t^{-\varepsilon}dt) = x^{1-\varepsilon}$. It is easy to see that $I_1 \geqslant Ax^{1-\varepsilon}\int_1^x t^{-1}Q(t)\,dt \to \infty$ as $x \to \infty$. Moreover, by Lemma 3, we have

$$\left|\int_1^x (dN - A\delta - A\,dt)*Q(t)\,dt\right| \leqslant \int_1^x Kxt^{-1}Q(x/t)Q(t)\,dt \leqslant c_2xQ(x).$$

It follows that

$$|I_2| \leqslant c_2\int_1^x xt^{-1}Q(x/t)(\delta + \varepsilon t^{-\varepsilon}dt) = c_2xQ(x) + c_2\varepsilon x\int_1^x t^{-\varepsilon-1}Q(x/t)\,dt.$$

For $\varepsilon > 0$ sufficiently small, $c_2\varepsilon < \frac{1}{3}A$. Also, for $x \geqslant 1/\varepsilon$, we have

$$\begin{aligned} \frac{1}{2}I_1 &= \frac{1}{2}Ax\int_1^x t^{-\varepsilon-1}Q(x/t)\,dt \geqslant \frac{1}{2}AxQ(x)\int_1^{1/\varepsilon} t^{-\varepsilon-1}\,dt \\ &\geqslant \frac{1}{2}AxQ(x)\varepsilon^\varepsilon\log\frac{1}{\varepsilon} \geqslant c_2xQ(x) \end{aligned}$$

since $\varepsilon^\varepsilon \geqslant \exp(-e^{-1})$. Therefore, $u_\varepsilon(x) \geqslant I_1 - |I_2| \geqslant \frac{1}{6}I_1$. □

LEMMA 5. *Suppose that* $N(x) = Ax + O(x\log^{-1}ex)$. *Then we have*

$$I = \int_1^x L\,dN*(\delta - \varepsilon t^{-\varepsilon}dt) = O(x).$$

PROOF. We have

$$I = \int_1^x L\,dN - \int_1^x L\,dN * \varepsilon t^{-\varepsilon} dt = I_1 - I_2,$$

say. By integration by parts,

$$I_1 = \int_1^x \log t\, dN(t) = N(x)\log x - \int_1^x N(t)t^{-1}\,dt = N(x)\log x + O(x).$$

Also, we have

$$\begin{aligned} I_2 &= \varepsilon \int_1^x L\,dN * t^{-\varepsilon} dt \\ &= \frac{\varepsilon}{1-\varepsilon}\int_1^x \left(\frac{x}{t}\right)^{1-\varepsilon} \log t\, dN(t) - \frac{\varepsilon}{1-\varepsilon}\int_1^x \log t\, dN(t) \\ &= \frac{\varepsilon}{1-\varepsilon} I_3 - \frac{\varepsilon}{1-\varepsilon} I_1, \end{aligned}$$

say. Again, by integration by parts,

$$\begin{aligned} I_3 &= \int_1^x x^{1-\varepsilon} t^{-1+\varepsilon} \log t\, dN(t) \\ &= N(x)\log x - x^{1-\varepsilon}\int_1^x N(t)\left\{(-1+\varepsilon)t^{-2+\varepsilon}\log t + t^{-2+\varepsilon}\right\} dt \\ &= N(x)\log x - x^{1-\varepsilon}(I_4 + I_5), \end{aligned}$$

say. It is easy to see that

$$\begin{aligned} I_4 &= (-1+\varepsilon)\int_1^x \left\{At + O\left(t\log^{-1} et\right)\right\} t^{-2+\varepsilon} \log t\, dt \\ &= (-1+\varepsilon)\left\{A\frac{x^\varepsilon}{\varepsilon}\log x + O_\varepsilon(x^\varepsilon)\right\}. \end{aligned}$$

Moreover,

$$I_5 = \int_1^x N(t)t^{-2+\varepsilon}\,dt = O\left(\int_1^x t^{-1+\varepsilon}\,dt\right) = O_\varepsilon(x^\varepsilon).$$

Combining all the estimates, we obtain $I = O(x)$. □

LEMMA 6. *Assume* (4), (5), (6), *and* (7). *Then*

$$\int_1^x L\,dN * (\delta - \varepsilon t^{-\varepsilon} dt) * Q(t)\,dt = O(x).$$

PROOF. By Lemma 2 and Lemma 5, the integral on the left-hand side equals

$$\int_1^x O(x/t)Q(t)\,dx = O\left(x\int_1^x t^{-1}Q(t)\,dt\right) = O(x). \qquad \square$$

We are now in the position to set up the upper estimate of the Theorem. The starting-point of the proof is Chebyshev's identity $d\psi * dN = L\,dN$ which, in this form, is still valid for Beurling's g-prime system. We convolve each side of it by $(\delta - \varepsilon t^{-\varepsilon}dt) * Q(t)dt$ and obtain

$$\int_1^x dN * (\delta - \varepsilon t^{-\varepsilon}dt) * Q(t)\,dt * d\psi = \int_1^x L\,dN * (\delta - \varepsilon t^{-\varepsilon}\,dt) * Q(t)\,dt.$$

Proof of the upper estimate. Assume (3), then (4), (5), (6), and (7) hold by Lemma 1. By Lemma 4 and Lemma 6,

$$\psi(x/B) \leqslant \int_1^x u_\varepsilon(x/t)\, d\psi(t) = O(x)$$

since $u_\varepsilon(x) \geqslant 1$ for $x \geqslant B$. □

To set up the lower estimate of the Theorem, we need one more lemma.

LEMMA 7. *Suppose that* $N(x) = Ax + o(x \log^{-1} ex)$. *Then*

$$I = \int_1^x L\, dN * (\delta - t^{-1}dt) = Ax + o(x).$$

PROOF. We have $I = \int_1^x L\, dN - \int_1^x L\, dN * t^{-1}\, dt = I_1 - I_2$, say. By integration by parts,

$$I_1 = \int_1^x \log t\, dN(t) = N(x)\log x - \int_1^x N(t) t^{-1}\, dt$$
$$= N(x)\log x - Ax + o\big(x \log^{-1} ex\big).$$

Moreover, we have

$$I_2 = \int_1^x \log(x/t) \log t\, dN(t)$$
$$= \log x \int_1^x \log t\, dN(t) - \int_1^x \log^2 t\, dN(t) = (\log x) I_1 - I_3,$$

say. Similarly, by integration by parts, $I_3 = N(x)\log^2 x - 2Ax(\log x - 1) + o(x)$. Combining the results, we arrive at

$$I = (1 - \log x) I_1 + I_3 = Ax + o(x). \quad \square$$

Proof of the lower estimate. Assume (3), then (4), (5), (6), and (7) hold by Lemma 1. We have

$$u_1(x) := \int_1^x dN * (\delta - t^{-1}dt) = N(x) - \int_1^x N(x/t) t^{-1}\, dt$$
$$= N(x) - \int_1^x \big\{ Axt^{-1} + O\big(xt^{-1}Q(x/t)\big)\big\} t^{-1}\, dt$$
$$= N(x) - Ax + A + O\Big(\int_1^x xt^{-2} Q(x/t)\, dt\Big)$$
$$= O\Big(1 + \int_1^x Q(t)\, dt\Big)$$

since $|N(x) - Ax| \leqslant xQ(x) \leqslant 2\int_1^x Q(t)\, dt$ for $x \geqslant 2$. From Chebyshev's identity, we have

$$\int_1^x dN * (\delta - t^{-1}\, dt) * d\psi = \int_1^x L\, dN * (\delta - t^{-1}\, dt).$$

By Lemma 2 and Lemma 7, the integral on the right-hand side is $Ax + o(x)$. Therefore, we have $\int_1^x u_1(x/t)\,d\psi(t) = Ax + o(x)$. We note that, by the upper estimate,

$$\begin{aligned}\int_1^x u_1(x/t)\,d\psi(t) &\leqslant K\int_1^x \left(1 + \int_1^{x/t} Q(u)\,du\right) d\psi(t)\\ &\leqslant K\psi(x) + K\int_1^x d\psi * Q(t)\,dt = K\psi(x) + K\int_1^x \psi(x/t)Q(t)\,dt\\ &\leqslant K\left\{\psi(x)\left(1 + \int_1^B Q(t)\,dt\right) + c_3 x\int_B^x t^{-1}Q(t)\,dt\right\}\\ &\leqslant c_4\psi(x) + c_3Kx\int_B^\infty t^{-1}Q(t)\,dt.\end{aligned}$$

For B sufficiently large, $c_3K\int_B^\infty t^{-1}Q(t)\,dt \leqslant \frac{1}{3}A$. Fixing B, for x sufficiently large, we have $c_4\psi(x) \geqslant \frac{1}{3}Ax$. This completes the proof of the Theorem. □

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