CHEBYSHEV TYPE ESTIMATES
FOR BEURLING GENERALIZED PRIME NUMBERS

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Abstract. We consider a Beurling generalized prime system for which the distribution function $N(x)$ of the integers satisfies

$$\int_1^\infty x^{-1} \left( \sup_{x \leq y} \left| \frac{N(y) - Ay}{y} \right| \right) \, dx < \infty$$

with constant $A > 0$. We shall prove that the Chebyshev type estimates

$$0 < \liminf_{x \to \infty} \frac{\psi(x)}{x}, \quad \limsup_{x \to \infty} \frac{\psi(x)}{x} < \infty$$

hold for the system. This gives a partial proof of one of Diamond’s conjectures.

Chebyshev was the first to establish the correct order of magnitude of the weighted counting function $\psi(x)$ of the ordinary prime numbers. He showed that there exist two numbers $\alpha > 0$ and $\beta < \infty$ such that

$$\liminf_{x \to \infty} \frac{\psi(x)}{x} \geq \alpha, \quad \limsup_{x \to \infty} \frac{\psi(x)}{x} \leq \beta.$$

The prime number theorem (P.N.T.) asserts that $\alpha = \beta = 1$. Here we shall study Chebyshev type estimates for Beurling generalized primes.

Let $\mathcal{P} = \{ p_i \}_{i=1}^\infty$, where $1 < p_1 \leq p_2 \leq \cdots, p_i \to \infty$, be a set of Beurling generalized (henceforth $g$-) prime numbers and $\mathcal{N} = \{ n_i \}_{i=0}^\infty$ be the associated set of $g$-integers (see [1, 2]). Define

$$N(x) = \sum_{n_i \leq x} 1, \quad \psi(x) = \sum_{p_i^j \leq x} \log p_i.$$

Beurling [2] proved that if

$$N(x) = Ax + O(x \log^{-\gamma} x)$$

for some constants $A > 0$ and $\gamma > 3/2$, then the P.N.T. holds for $\mathcal{P}$. If $\gamma = 3/2$ in (2), the P.N.T. need not hold as Diamond [3] showed by an example based on a continuous example of Beurling. Diamond [4] also showed that if $\gamma > 1$ in (2) then (1) holds. On the other hand, (1) is not generally true if $\gamma < 1$ in (2), as an example
\[ \int_1^\infty x^{-2} |N(x) - Ax| \, dx < \infty \]
is sufficient to imply Chebyshev type bounds for a g-prime system.

In the present paper, we shall prove the following

**Theorem.** If the distribution function \( N(x) \) of the g-integers of a g-prime system satisfies
\[ \int_1^\infty x^{-1} \left( \sup_{x \leq y} \left| \frac{N(y) - Ay}{y} \right| \right) \, dx < \infty \]
with constant \( A > 0 \), then there exist numbers \( \alpha > 0 \) and \( \beta < \infty \) for which (1) holds.

This theorem gives a partial proof of Diamond's conjecture.

We divide the proof of the Theorem into several lemmas.

**Lemma 1.** Assume (3). Then there exists a function \( Q(x) \) such that
\[ Q(x) \text{ is nonincreasing,} \]
\[ \int_1^\infty Q(x) x^{-1} \, dx < \infty, \]
\[ Q(x) \leq 4Q(x^2) \quad \text{for all } x \geq 1, \]
and
\[ Q(x) \geq \sup_{x \leq y} \left| \frac{N(y) - Ay}{y} \right|. \]

**Proof.** Let
\[ Q_1(x) = \sup_{x \leq y} \left| \frac{N(y) - Ay}{y} \right|. \]

We note that \( Q_1(x) \) is nonnegative and nonincreasing. Define \( Q(x) \) recursively by setting
\[ Q(x) = \begin{cases} Q_1(1) & \text{for } 1 \leq x \leq 2, \\ \max \left\{ Q_1(2^{2m-1}), 4^{-1}Q(2^{2m-1}) \right\} & \text{for } 2^{2m-1} < x \leq 2^{2m}, \ m \in \mathbb{N}. \end{cases} \]

We now verify that this function satisfies each of the following conditions:
(i) \( Q(x) \geq Q_1(x) \) (this is obvious).
(ii) \( Q(x) \downarrow. \)
We note that \( Q(x) \) is constant on \( 2^{2m-1} < x \leq 2^{2m} \). If \( 1 \leq x_1 < 2 < x_2 \leq 2^2 \), then
\[ Q(x_2) = \max \left\{ Q_1(2), 4^{-1}Q(2) \right\} \leq Q_1(1) = Q(x_1). \]
If \( 2^{2m-1} < x_1 \leq 2^{2m} \), then
\[ Q(x_2) = \max \left\{ Q_1(2^{2m}), 4^{-1}Q(2^{2m}) \right\} \leq Q(2^{2m}) = Q(x_1) \]
since, by (i), \( Q_1(2^{2m}) \leq Q(2^{2m}). \)
(iii) \( Q(x) \leq 4Q(x^2). \)
For $1 \leq x \leq 2$, we have $1 \leq x^2 \leq 2^2$. If $1 \leq x^2 \leq 2$, we have nothing to show. If $2 < x^2 \leq 2^2$, we have
$Q(x^2) \geq 4^{-1}Q(2) = 4^{-1}Q(x)$.
For $2^{m-1} < x \leq 2^m$, $m \geq 1$, we have $2^m < x^2 \leq 2^{m+1}$ and hence
$Q(x^2) \geq 4^{-1}Q(2^m) = 4^{-1}Q(x)$.
(iv) $\int_1^\infty Q(x) x^{-1} \, dx < \infty$.
We have to show that
$$\sum_{m=1}^{\infty} \int_{2^{m-1}}^{2^m} Q(x) x^{-1} \, dx < \infty.$$  
There are two distinct cases that we have to consider separately.

Case I. $4^{-1}Q(2^m) \geq Q_l(2^m)$ for all $m \geq m_0$. In this case, for $2^{m_0} < x \leq 2^{m_0+1}$,
$Q(x) = \max\{Q_l(2^{m_0}), 4^{-1}Q(2^{m_0})\} = 4^{-1}Q(2^{m_0})$.
By induction, for all $m \geq m_0 + 1$, $2^{m-1} < x \leq 2^m$, we have
$Q(x) = 4^{-1}Q(2^m) = 4^{-1}(m-m_0)Q(2^{m_0})$.
Therefore,
$$\sum_{m=m_0+1}^{\infty} \int_{2^{m-1}}^{2^m} Q(x) x^{-1} \, dx = \sum_{m=m_0+1}^{\infty} 4^{-1}(m-m_0)Q(2^{m_0}) \int_{2^{m-1}}^{2^m} x^{-1} \, dx$$
$$= \sum_{m=m_0+1}^{\infty} 4^{-1}(m-m_0)Q(2^{m_0}) \log(2^{m-1}) = Q(2^{m_0}) \log(2^{m_0+1}) .$$

Case II. There exist $m_1 < m_2 < \cdots < m_k < m_{k+1} < \cdots$ such that $Q_l(2^{m_1}) > 4^{-1}Q(2^{m_1})$ and $Q_l(2^{m_2}) \leq 4^{-1}Q(2^{m_2}) \forall m \neq m_k$. In this case, we can show that
$$\sum_{m=m_1+1}^{m_{k+1}} \int_{2^{m-1}}^{2^m} Q(x) x^{-1} \, dx \leq 4Q_l(2^{m_1}) \log(2^{m_1-1}) .$$
Actually, if $m_{k+1} = m_k + 1$, the left-hand side equals
$$\int_{2^{m_k+1}}^{2^{m_k+1}} Q(x) x^{-1} \, dx = Q_l(2^{m_k}) \int_{2^{m_k}}^{2^{m_{k+1}}} x^{-1} \, dx$$
$$= Q_l(2^{m_k}) \log(2^{m_k}) = 2Q_l(2^{m_k}) \log(2^{m_k-1}) .$$
Therefore, we consider $m_k < m_k + 2 \leq m_{k+1}$. For $2^{m_k} < x \leq 2^{m_{k+1}}$, we have
$Q(x) = \max\{Q_l(2^{m_k}), 4^{-1}Q(2^{m_k})\} = Q_l(2^{m_k})$.
Then, for $2^{m_k+1} < x \leq 2^{m_{k+2}}$,
$Q(x) = 4^{-1}Q(2^{m_k+1}) = 4^{-1}Q_l(2^{m_k})$.
By induction, for all \( m \in [m_k + 1, m_{k+1}] \) and \( 2^{2m-1} < x \leq 2^{2^m} \), we have \( Q(x) = 4^{-(m-m_k)+1}Q_1(2^{2^m}) \). Therefore,

\[
\sum_{m=m_k+1}^{m_{k+1}} \int_{2^{2m-1}}^{2^{2m}} Q(x)x^{-1}dx = \sum_{m=m_k+1}^{m_{k+1}} 4^{-(m-m_k)+1}Q_1(2^{2^m}) \int_{2^{2m-1}}^{2^{2m}} x^{-1}dx = 4^{-(m-m_k)+1}Q_1(2^{2^m}) \log(2^{2m-1})
\]

\[
= Q_1(2^{2^m}) \log(2^{2^m}) \sum_{k=1}^{m_{k+1}-m_k} 2^{-k} \leq 4Q_1(2^{2^m}) \log(2^{2^m-1}).
\]

Thus we have

\[
\sum_{m=m_k+1}^{m_{k+1}} \int_{2^{2m-1}}^{2^{2m}} Q(x)x^{-1}dx = \sum_{k=1}^{m_{k+1}-m_k} 4Q_1(2^{2^m}) \log(2^{2m-1}) \leq 4 \sum_{k=1}^{m_{k+1}-m_k} \int_{2^{2m-1}}^{2^{2m}} x^{-1}dx.
\]

To show the last sum is finite, we note that

\[
\int_{2^{2m-1}}^{2^{2m}} Q_1(x)x^{-1}dx = \int_{2^{2m-1}}^{2^{2m}} x^{-1}dx = \log(2^{2m-1}) \leq \frac{4}{2} \int_{2^{2m-1}}^{2^{2m}} x^{-1}dx < \infty
\]

by (3). This completes the proof of Lemma 1.

**LEMMA 2.** Assume (4) and (5). Then \( Q(x) = o(\log^{-1} ex) \).

**Proof.** Given \( \epsilon > 0 \), for \( x > x_0 \), we have \( \int_{x_0}^{\infty} Q(t)t^{-1}dt < \epsilon/2 \). Thus, for \( x > x_0^\frac{1}{\epsilon} \),

\[
Q(x) \log x < \int_{\sqrt{x}}^{x} Q(t)t^{-1}dt < \epsilon/2,
\]

i.e., \( Q(x) \log x < \epsilon \). □

The following lemma shows that \( Q(x)dx \) has a kind of “stability” under multiplicative convolution.

**LEMMA 3.** Assume (4), (5), and (6). Then \( \int_1^x t^{-1}Q(x/t)Q(t)dt \leq c_1Q(x) \), where \( c_1 \) is a constant.

**Proof.** We have

\[
\int_1^x t^{-1}Q(x/t)Q(t)dt = 2 \int_{1}^{\sqrt{x}} t^{-1}Q(x/t)Q(t)dt \leq 2Q(\sqrt{x}) \int_{1}^{\sqrt{x}} Q(t)t^{-1}dt \leq c_1Q(x)
\]

since, by (4) and (6), \( Q(x/t) \leq Q(\sqrt{x}) \leq 4Q(x) \) for \( 1 \leq t \leq \sqrt{x} \). □
By using the "stability" of $Q(x) \, dx$, we can prove the following lemma which is the main step in the proof of the Theorem.

**Lemma 4.** Assume (4), (5), (6), and (7). Then, for fixed and sufficiently small $\varepsilon > 0$, we have

$$u_\varepsilon(x) := \int_1^x dN * (\delta - \varepsilon t \, dt) * Q(t) \, dt \geq 0$$

for $x \geq 1$ and $u_\varepsilon(x) \to \infty$ as $x \to \infty$.

**Proof.** If $1 \leq x \leq 1/\varepsilon$, then

$$\int_1^x dN * (\delta - \varepsilon t \, dt) \geq N(x) - \varepsilon N(x)(x - 1) \geq 0.$$  

The lemma is certainly true for $1 \leq x \leq 1/\varepsilon$, since the third convolution factor is everywhere nonnegative.

If $x > 1/\varepsilon$, we utilize all the convolution factors. We write

$$u_\varepsilon(x) = A \int_1^x (\delta + dt) * (\delta - \varepsilon t \, dt) * Q(t) \, dt$$

$$+ \int_1^x (dN - A\delta - Adt) * (\delta - \varepsilon t \, dt) * Q(t) \, dt$$

$$= I_1 + I_2,$$

say. We will show that $I_1$ is positive, that $I_1 \to \infty$ as $x \to \infty$ and that $I_2$ is negligible. Actually, we have

$$I_1 = A \int_1^x \frac{x}{u} t^{-1} Q(t) \, dt = Ax \int_1^x u^{-1} Q(x/u) \, du$$

since $\int_1^x (\delta + dt) * (\delta - \varepsilon t \, dt) = x^{1-\varepsilon}$. It is easy to see that $I_1 \geq Ax^{1-\varepsilon} \int_1^x t^{-1} Q(t) \, dt \to \infty$ as $x \to \infty$. Moreover, by Lemma 3, we have

$$\left| \int_1^x (dN - A\delta - Adt) * Q(t) \, dt \right| \leq \int_1^x Kxt^{-1} Q(x/t) Q(t) \, dt \leq c_2 x Q(x).$$

It follows that

$$|I_2| \leq c_2 \int_1^x xt^{-1} Q(x/t)(\delta + \varepsilon t^{-1} dt) = c_2 x Q(x) + c_2 \varepsilon \int_1^x t^{-\varepsilon-1} Q(x/t) \, dt.$$  

For $\varepsilon > 0$ sufficiently small, $c_2 \varepsilon < \frac{1}{3} A$. Also, for $x \geq 1/\varepsilon$, we have

$$\frac{1}{2} I_1 = \frac{1}{2} Ax \int_1^x t^{-\varepsilon-1} Q(x/t) \, dt \geq \frac{1}{2} Ax Q(x) \int_1^{1/\varepsilon} t^{-1} \, dt$$

$$\geq \frac{1}{2} Ax Q(x) \varepsilon \log \frac{1}{\varepsilon} \geq c_2 x Q(x)$$

since $\varepsilon \varepsilon > \exp(-\varepsilon^{-1})$. Therefore, $u_\varepsilon(x) \geq I_1 - |I_2| \geq \frac{1}{6} I_1$. \qed

**Lemma 5.** Suppose that $N(x) = Ax + O(x \log^{-1} x)$. Then we have

$$I = \int_1^x LdN * (\delta - \varepsilon t^{-1} dt) = O(x).$$
Proof. We have

\[ I = \int_1^X L \, dN - \int_1^X L \, dN \ast e^{-\varepsilon t} dt = I_1 - I_2, \]

say. By integration by parts,

\[ I_1 = \int_1^X \log t \, dN(t) = N(x) \log x - \int_1^X N(t) t^{-1} \, dt = N(x) \log x + O(x). \]

Also, we have

\[ I_2 = \varepsilon \int_1^X L \, dN \ast t^{-\varepsilon} dt = \frac{\varepsilon}{1 - \varepsilon} \int_1^X \left( \frac{x}{t} \right)^{1-\varepsilon} \log t \, dN(t) - \frac{\varepsilon}{1 - \varepsilon} \int_1^X \log t \, dN(t) \]

say. Again, by integration by parts,

\[ I_3 = \int_1^X x^{1-\varepsilon} t^{-1+\varepsilon} \log t \, dN(t) \]

\[ = N(x) \log x - x^{1-\varepsilon} \int_1^X N(t) \left\{ (-1 + \varepsilon) t^{-2+\varepsilon} \log t + t^{-2+\varepsilon} \right\} \, dt \]

\[ = N(x) \log x - x^{1-\varepsilon} (I_4 + I_5), \]

say. It is easy to see that

\[ I_4 = (-1 + \varepsilon) \int_1^X \left\{ At + O(t \log^{-1} et \right\} t^{-2+\varepsilon} \log t \, dt \]

\[ = (-1 + \varepsilon) \left\{ A \frac{x^{1-\varepsilon}}{\varepsilon} \log x + O(x) \right\}. \]

Moreover,

\[ I_5 = \int_1^X N(t) t^{-2+\varepsilon} \, dt = O\left( \int_1^X t^{-1+\varepsilon} \, dt \right) = O(x^{\varepsilon}). \]

Combining all the estimates, we obtain \( I = O(x). \) □

Lemma 6. Assume (4), (5), (6), and (7). Then

\[ \int_1^X L \, dN \ast (\delta - e^{-\varepsilon} t) \, Q(t) \, dt = O(x). \]

Proof. By Lemma 2 and Lemma 5, the integral on the left-hand side equals

\[ \int_1^X O(x/t) Q(t) \, dx = O\left( x \int_1^X t^{-1} Q(t) \, dt \right) = O(x). \] □

We are now in the position to set up the upper estimate of the Theorem. The starting-point of the proof is Chebyshev's identity \( d\psi \ast dN = L \, dN \) which, in this form, is still valid for Beurling's g-prime system. We convolve each side of it by \((\delta - e^{-\varepsilon} t) \, Q(t) \, dt\) and obtain

\[ \int_1^X dN \ast (\delta - e^{-\varepsilon} t) \ast Q(t) \, dt \ast d\psi = \int_1^X L \, dN \ast (\delta - e^{-\varepsilon} t) \ast Q(t) \, dt. \]
Proof of the upper estimate. Assume (3), then (4), (5), (6), and (7) hold by Lemma 1. By Lemma 4 and Lemma 6,

\[ \psi (x/B) \leq \int_1^x u_s(x/t) \, d\psi (t) = O(x) \]

since \( u_s(x) \geq 1 \) for \( x \geq B \). \( \square \)

To set up the lower estimate of the Theorem, we need one more lemma.

**Lemma 7.** Suppose that \( N(x) = Ax + o(x \log^{-1} e^x) \). Then

\[ I = \int_1^x L \, dN \ast (\delta - t^{-1} \, dt) = Ax + o(x). \]

**Proof.** We have \( I = \int_1^x L \, dN - \int_1^x L \, dN \ast t^{-1} \, dt = I_1 - I_2, \) say. By integration by parts,

\[ I_1 = \int_1^x \log t \, dN(t) = N(x) \log x - \int_1^x N(t) \, t^{-1} \, dt = N(x) \log x - Ax + o(x \log^{-1} e^x). \]

Moreover, we have

\[ I_2 = \int_1^x \log(x/t) \log t \, dN(t) \]

\[ = \log x \int_1^x \log t \, dN(t) - \int_1^x \log^2 t \, dN(t) = (\log x) I_1 - I_3, \]

say. Similarly, by integration by parts, \( I_3 = N(x) \log^2 x - 2Ax(\log x - 1) + o(x) \). Combining the results, we arrive at

\[ I = (1 - \log x) I_1 + I_3 = Ax + o(x). \] \( \square \)

Proof of the lower estimate. Assume (3), then (4), (5), (6), and (7) hold by Lemma 1. We have

\[ u_1(x) := \int_1^x dN \ast (\delta - t^{-1} \, dt) = N(x) - \int_1^x N(x/t) \, t^{-1} \, dt \]

\[ = N(x) - \int_1^x \left\{ Ax \, t^{-1} + O(xt^{-1}Q(x/t)) \right\} t^{-1} \, dt \]

\[ = N(x) - Ax + A + O \left( \int_1^x xt^{-2}Q(x/t) \, dt \right) \]

\[ = O \left( 1 + \int_1^x Q(t) \, dt \right) \]

since \( |N(x) - Ax| \leq xQ(x) \leq 2 \int_1^x Q(t) \, dt \) for \( x \geq 2 \). From Chebyshev's identity, we have

\[ \int_1^x dN \ast (\delta - t^{-1} \, dt) \ast d\psi = \int_1^x L \, dN \ast (\delta - t^{-1} \, dt). \]
By Lemma 2 and Lemma 7, the integral on the right-hand side is $Ax + o(x)$. Therefore, we have $\int_1^x u_1(x/t)\,d\psi(t) = Ax + o(x)$. We note that, by the upper estimate,

$$\int_1^x u_1(x/t)\,d\psi(t) \leq K \int_1^x \left(1 + \int_1^{x/t} Q(u)\,du\right)\,d\psi(t)$$

$$\leq K\psi(x) + K \int_1^x d\psi \ast Q(t)\,dt = K\psi(x) + K \int_1^x \psi(x/t)Q(t)\,dt$$

$$\leq K \left\{ \psi(x) \left[ 1 + \int_1^B Q(t)\,dt \right] + c_3 \int_B^\infty t^{-1}Q(t)\,dt \right\}$$

$$\leq c_4\psi(x) + c_3Kx \int_B^\infty t^{-1}Q(t)\,dt.$$

For $B$ sufficiently large, $c_3K\int_B^\infty t^{-1}Q(t)\,dt \leq \frac{1}{3}A$. Fixing $B$, for $x$ sufficiently large, we have $c_4\psi(x) \geq \frac{1}{3}Ax$. This completes the proof of the Theorem. ☐

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