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# DIHEDRAL ALGEBRAS ARE CYCLIC

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ABSTRACT. This note gives a simple proof of the following theorem of Rowen and Saltman: *Every central simple algebra split by a Galois extension of rank* $2n$ *(*$n$ *odd) with dihedral Galois group is cyclic if the center contains a primitive nth root of unity.*

The aim of this note is to provide a short conceptual proof of the following theorem of Rowen and Saltman [**3**].

THEOREM. *Let* $n$ *be an odd (positive) integer and let* $F$ *be a field containing a primitive nth root of unity. Every central simple F-algebra split by a Galois extension of* $F$ *of rank* $2n$ *with dihedral Galois group is also split by a cyclic extension of* $F$.

The proof given here only uses basic properties of symbols and of the corestriction map, and can be adapted to the case where $\operatorname{char} F = n$ (instead of $F$ containing a primitive $n$th root of unity), to yield a particular case of a general theorem of Albert [**1**].

Henceforth, we fix an odd integer $n$ and a field $F$ containing a primitive $n$th root of unity, and a Galois extension $K/F$ with dihedral Galois group generated by two elements $\sigma$, $\tau$ subject to the relations

$$\sigma^n = 1, \qquad \tau^2 = 1, \qquad \sigma\tau\sigma = \tau.$$

Let $L$ be the fixed field of $\sigma$ in $K$.

LEMMA. *There is an element* $a \in L^\times$ *such that* $K = L(a^{1/n})$ *and* $N_{L/F}(a) \in F^{\times n}$.

PROOF. Since $K/L$ is cyclic of rank $n$ and $L$ contains a primitive $n$th root of unity $\zeta$, one can find $\alpha \in K$ such that $K = L(\alpha)$ and $\sigma(\alpha) = \zeta\alpha$. Applying $\sigma\tau$ to both sides of this equation yields $\tau(\alpha) = \zeta\sigma\tau(\alpha)$, and it follows that $\alpha\tau(\alpha)$ is fixed under $\sigma$. This element is clearly fixed under $\tau$ too, so $\alpha\tau(\alpha) \in F^\times$. Denoting $\alpha^n = a$, we have $K = L(a^{1/n})$ and $N_{L/F}(a) = (\alpha\tau(\alpha))^n \in F^{\times n}$, as required. Q.E.D.

PROOF OF THE THEOREM. Let $A$ be a central simple $F$-algebra split by $K$. By [**2**, Theorem 14, p. 68], $A$ decomposes as $A_1 \otimes_F A_2$ where the degree of $A_1$ is a power of 2 and the degree of $A_2$ is odd. Both $A_1$ and $A_2$ are split by $K$, hence $A_1$ is split by $L$, and it suffices to prove that $A_2$ is split by a cyclic extension of $F$.

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Since the degree of $A_2$ is odd, $A_2$ is similar (in the Brauer group of $F$) to an even power of itself: let $A_2 \sim A_2^{2m}$ for some integer $m$. By [**2**, Lemma 9, p. 54], $A_2^2 \sim \mathrm{Cor}_{L/F}(A_2 \otimes_F L)$, hence raising both sides to the $m$th power, we get

$$A_2 \sim \mathrm{Cor}_{L/F}(A_2^m \otimes_F L).$$

Now, since $K = L(a^{1/n})$ splits $A_2$, hence also $A_2^m \otimes L$, there exists $b \in L^\times$ such that $A_2^m \otimes L$ is similar to the symbol algebra $(a, b)$ of degree $n$ over $L$ (denoted by $(a, b; n, L, \zeta)$ in [**2**]; see [**2**, Lemma 1, p. 78]), hence

$$A_2 \sim \mathrm{Cor}_{L/F}(a, b).$$

We complete the proof by showing that the corestriction of $(a, b)$ is a symbol algebra: this readily follows from the "projection formula" [**2**, Theorem 7, p. 88] if $b \in F$, so we can assume $b \notin F$. (Note that $a \notin F$, or else the lemma would imply $a \in F^{\times n}$, a contradiction.) Since $[L:F] = 2$, one can then find $a', b' \in F$, both nonzero, such that $aa' + bb' = 0$ or 1. Then $(aa', bb') \sim 1$, so that

$$(a, b) \sim (a, b')^{-1} \otimes (a', bb')^{-1}.$$

Taking the corestriction of both sides, we get by the "projection formula":

$$\mathrm{Cor}_{L/F}(a, b) \sim \left(N_{L/F}(a), b'\right)^{-1} \otimes \left(a', N_{L/F}(bb')\right)^{-1}.$$

The lemma shows that the first factor on the right-hand side is trivial, hence $\mathrm{Cor}_{L/F}(a, b)$ is similar to a symbol algebra. Q.E.D.

REMARK. This proof can be readily adapted to the case where $\mathrm{char}\, F = n$ (prime), by replacing symbols $(a, b)$ by $n$-symbols $[a, b)$: one first shows that $K = L(\alpha)$ for some $\alpha$ such that $a := \alpha^n - \alpha \in L$ and $\mathrm{Tr}_{L/F}(a) = u^n - u$ for some $u \in F$; the same arguments as above then show that it suffices to prove that $\mathrm{Cor}_{L/F}[a, b)$ is a symbol algebra (for any $b \in L$), and this follows from a decomposition:

$$\mathrm{Cor}_{L/F}[a, b) \sim \left[\mathrm{Tr}_{L/F}(a), b'\right) \otimes \left[a', N_{L/F}(b'')\right).$$

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