PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 2, October 1987

A CHARACTERIZATION OF INNER AUTOMORPHISMS

PAUL E. SCHUPP

ABSTRACT. It turns out that one can characterize inner automorphisms without mentioning either conjugation or specific elements. We prove the following

THEOREM *Let G be a group and let α be an automorphism of G. The automorphism α is an inner automorphism of G if and only if α has the property that whenever G is embedded in a group H, then α extends to some automorphism of H.*

It turns out that in the category of groups one can characterize inner automorphisms without mentioning either conjugation or specific elements. It is obvious that if G is a group with α an inner automorphism of G and G is embedded in a group H, then α extends to an automorphism of H; viz., conjugation by the correct element. Angus Macintyre asked whether or not this extension property actually characterizes inner automorphisms. In this note we prove that it does.

THEOREM. *Let G be a group and let α be an automorphism of G. The automorphism α is an inner automorphism of G if and only if α has the property that whenever G is embedded in a group H then α extends to some automorphism of H.*

PROOF. A subgroup K of a group H is *malnormal* in H if $hKh^{-1} \cap K = \{1\}$ for all $h \in H \backslash K$. We shall prove that any group G is embeddable as a malnormal subgroup of a complete group H.

This establishes the theorem, for suppose that the automorphism α of G extends to an automorphism β of H where G and H are as in the previous paragraph. Since H is complete, β must be an inner automorphism of H, say that $\beta(h) = h_0 h h_0^{-1}$. Since β extends α, we have $h_0 G h_0^{-1} = G$ but since G is malnormal in H, we have $h_0 \in G$ (except possibly in the uninteresting case $G = \{1\}$) and thus α is an inner automorphism of G.

The theorem in the case where G is countable is really already proved in Miller and Schupp [**1**]. Here we follow the same idea of using small cancellation products but arrange things to work when the cardinality of G is arbitrary. We shall use only a few well-known results of small cancellation theory (see [**2**]). We shall construct a certain small cancellation product of G and sufficiently many finite cyclic groups. We use σ, τ, and λ to denote ordinal numbers.

Let $\{g_\sigma : \sigma < \lambda\}$ be a well-ordered set of generators for G. Let

$$F = G * \langle x; x^{11}\rangle * \left(\mathop{*}_{\sigma<\lambda} \langle b_\sigma; b_\sigma^7\rangle \right).$$

For $\sigma < \lambda$, let

$$r_\sigma = g_\sigma (xb_\sigma) x b_\sigma^2 (xb_\sigma)^2 x b_\sigma^2 \cdots (xb_\sigma)^{80}$$

Received by the editors June 13, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20E36.

This research was supported by National Science Foundation Grant MCS-80-03251.

and let

$$s_\sigma = xb_\sigma^2(xb_\sigma)^{81}xb_\sigma^2\cdots(xb_\sigma)^{160}.$$

For $\sigma < \tau < \lambda$ let

$$t_{\sigma,\tau} = (xb_\sigma)xb_\tau(xb_\sigma)^2xb_\tau\cdots(xb_\sigma)^{80}.$$

Let R be the symmetrized subset of F generated by $\{r_\sigma, s_\sigma, t_{\sigma,\tau}\colon \sigma < \tau < \lambda\}$. It is clear that R satisfies the small cancellation condition $C'(1/10)$. Let N be the normal closure of R in F and let $H = F/N$. Then G is embedded in H. Now no $r \in R$ has a subword of the form ufu^{-1} with $u \neq 1$ and f in a factor of F. If u does not end with a letter from the same factor as f, then ufu^{-1} is in free product normal form as written. If u is R-reduced, then $ufu^{-1}f' \neq 1$ in H for any f' in a factor of F since small cancellation theory says that any nontrivial word of F which represents the identity of G must contain more than seven-tenths of an element of R. Thus G is malnormal in H. This argument also shows that H has trivial center.

We must show that H is complete. Clearly, H is generated by x and the b_σ in view of the relators r_σ. We next note that none of x or the b_η is contained in the subgroup of H generated by the other generators and G. For suppose that some $b_\eta \in \mathrm{Gp}\{G, x, b_\sigma\colon \sigma \neq \eta\}$. Then an equation $b_\eta = w$ holds in H where w does not contain b_η. We may suppose that w is R-reduced. Now any element of R contains only one b_σ generator or many occurrences of both b_σ and b_η. Thus the equation $wb_\eta^{-1} = 1$ in H cannot hold.

The Torsion Theorem for small cancellation quotients of free products says that the only elements of finite order in H are conjugates of elements of G, conjugates of powers of x, and conjugates of powers of the b_σ. Knowing all the elements of finite order is a key point in analyzing automorphisms of H. Let φ be any automorphism of H. We need only prove that, up to an inner automorphism, φ fixes x and all the b_σ. Since $\varphi(x)$ has order eleven, $\varphi(x)$ is either a conjugate of an element of G or a conjugate of a power of x. Following φ by an inner automorphism we may suppose that $\varphi(x)$ is in a factor. First suppose that $\varphi(x) = g \in G$. Now $H = \mathrm{Gp}\{\varphi(x), \varphi(b_\sigma)\colon \sigma < \lambda\}$ but, by the remark above, $x \notin \mathrm{Gp}\{G, b_\sigma\colon \sigma < \lambda\}$. Thus for some η we must have $\varphi(b_\eta) = ug_1u^{-1}$ with $g_1 \in G$ or $\varphi(b_\eta) = ub_\gamma^k u^{-1}$ where u is R-reduced and contains x. Following φ by another inner automorphism if necessary, we may assume that u does not begin with an element of G. In the case that $\varphi(b_\eta) = ub_\gamma^k u^{-1}$ we have $\varphi(s_\eta) = gub_\gamma^{2k}u^{-1}\cdots(gub_\gamma^k u^{-1})^{160}$ which must be equal to the identity in H. But the above expression is in free product normal form as written and cannot contain seven-tenths of an element of R. Similarly, the case $\varphi(b_\eta) = ug_1u^{-1}$ leads to a contradiction. We conclude that $\varphi(x) = x^j$ for some j. With this fact a similar argument now establishes that each $\varphi(b_\sigma) = b_\tau^k$ for some choice of τ and k. But now application of φ to the relators s_σ shows that j and each k must be equal to one. For,

$$\varepsilon(s_\sigma) = x^j b_\gamma^{2k}(x^j b_\gamma^k)^{80}\cdots(x^j b_\gamma^k)^{160}$$

and the only way that the latter expression can contain a large part of a relator is to have $j = k = 1$. So we conclude that φ fixes x and permutes the b_σ.

We now show that φ fixes the b_σ. First of all, we show that if $\sigma < \tau$, $\varphi(b_\sigma) = b_\gamma$, and $\varphi(b_\tau) = b_\eta$ then $\gamma < \eta$, that is, φ is order-preserving on subscripts. For, applying φ to $t_{\sigma,\tau}$ we have

$$\varphi(t_{\sigma,\tau}) = xb_\gamma xb_\eta(xb_\gamma)^2\cdots(xb_\gamma)^{80}$$

which must equal the indentity in H. The only element of R which $\varphi(t_{\sigma,\tau})$ could contain a large part of is $t_{\gamma,\eta}$ which yields $\gamma < \eta$. But now we can conclude the proof, for, by the remark on the minimality of the generating set $\{x, b_\sigma : \sigma < \lambda\}$, all b_σ must occur in the range of φ. If φ does not fix every b_σ, let δ be the least subscript such that $\varphi(b_\delta) \neq b_\delta$. Then $\varphi(b_\delta) = b_\eta$ with $\eta > \delta$. Since φ is order-preserving on subscripts, we have $b_\delta \notin \mathrm{Gp}\{x, \varphi(b_\sigma) : \sigma < \lambda\}$ contradicting the fact that the latter set generates H. Thus H is complete. □

BIBLIOGRAPHY

1. C. F. Miller and P. E. Schupp, *Embeddings into Hopfian groups*, J. Algebra **17** (1971), 171–176.
2. P. E. Schupp, *A survey of small cancellation theory*, Word Problems and the Burnside Problem (Boone, Cannonito, Lyndon eds.), North-Holland, 1972, pp. 569–589.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801

LITP, UNIVERSITÉ PARIS 7, PARIS, FRANCE