A CHARACTERIZATION OF INNER AUTOMORPHISMS

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ABSTRACT. It turns out that one can characterize inner automorphisms without mentioning either conjugation or specific elements. We prove the following theorem:

Theorem. Let $G$ be a group and let $\alpha$ be an automorphism of $G$. The automorphism $\alpha$ is an inner automorphism of $G$ if and only if $\alpha$ has the property that whenever $G$ is embedded in a group $H$, then $\alpha$ extends to some automorphism of $H$.

It turns out that in the category of groups one can characterize inner automorphisms without mentioning either conjugation or specific elements. It is obvious that if $G$ is a group with $\alpha$ an inner automorphism of $G$ and $G$ is embedded in a group $H$, then $\alpha$ extends to an automorphism of $H$; viz., conjugation by the correct element. Angus Macintyre asked whether or not this extension property actually characterizes inner automorphisms. In this note we prove that it does.

Theorem. Let $G$ be a group and let $\alpha$ be an automorphism of $G$. The automorphism $\alpha$ is an inner automorphism of $G$ if and only if $\alpha$ has the property that whenever $G$ is embedded in a group $H$ then $\alpha$ extends to some automorphism of $H$.

Proof. A subgroup $K$ of a group $H$ is malnormal in $H$ if $hKh^{-1} \cap K = \{1\}$ for all $h \in H \setminus K$. We shall prove that any group $G$ is embeddable as a malnormal subgroup of a complete group $H$.

This establishes the theorem, for suppose that the automorphism $\alpha$ of $G$ extends to an automorphism $\beta$ of $H$ where $G$ and $H$ are as in the previous paragraph. Since $H$ is complete, $\beta$ must be an inner automorphism of $H$, say that $\beta(h) = h_0h_0^{-1}$.

Since $\beta$ extends $\alpha$, we have $h_0Gh_0^{-1} = G$ but since $G$ is malnormal in $H$, we have $h_0 \not\in G$ (except possibly in the uninteresting case $G = \{1\}$) and thus $\alpha$ is an inner automorphism of $G$.

The theorem in the case where $G$ is countable is really already proved in Miller and Schupp [1]. Here we follow the same idea of using small cancellation products but arrange things to work when the cardinality of $G$ is arbitrary. We shall use only a few well-known results of small cancellation theory (see [2]). We shall construct a certain small cancellation product of $G$ and sufficiently many finite cyclic groups. We use $\sigma, \tau$, and $\lambda$ to denote ordinal numbers.

Let $\{g_\sigma : \sigma < \lambda\}$ be a well-ordered set of generators for $G$. Let

$$F = G * \langle x; x^{11} \rangle * \left( \bigodot_{\sigma < \lambda} \langle b_\sigma, b_\sigma^2 \rangle \right).$$

For $\sigma < \lambda$, let

$$r_\sigma = g_\sigma(xb_\sigma)xb_\sigma^2(xb_\sigma)^2xb_\sigma^2 \cdots (xb_\sigma)^{80}$$

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and let
\[ s_\sigma = x b_\sigma^2 (x b_\sigma)^{61} x b_\sigma^2 \cdots (x b_\sigma)^{160}. \]
For \( \sigma < \tau < \lambda \) let
\[ t_{\sigma, \tau} = (x b_\sigma) x b_\tau (x b_\tau)^3 x b_\tau \cdots (x b_\tau)^{80}. \]
Let \( R \) be the symmetrized subset of \( F \) generated by \( \{t_{\sigma, \tau}, s_\sigma, t_{\sigma, \tau}: \sigma < \tau < \lambda \} \). It is clear that \( R \) satisfies the small cancellation condition \( C'(1/10) \). Let \( N \) be the normal closure of \( R \) in \( F \) and let \( H = F/N \). Then \( G \) is embedded in \( H \). Now no \( r \in R \) has a subword of the form \( u f u^{-1} \) with \( u \neq 1 \) and \( f \) in a factor of \( F \). If \( u \) does not end with a letter from the same factor as \( f \), then \( u f u^{-1} \) is in free product normal form as written. If \( u \) is \( R \)-reduced, then \( u f u^{-1} f' \neq 1 \) in \( H \) for any \( f' \) in a factor of \( F \) since small cancellation theory says that any nontrivial word of \( F \) which represents the identity of \( G \) must contain more than seven-tenths of an element of \( R \). Thus \( G \) is malnormal in \( H \). This argument also shows that \( H \) has trivial center.

We must show that \( H \) is complete. Clearly, \( H \) is generated by \( x \) and the \( b_\sigma \) in view of the relators \( r_\sigma \). We next note that none of \( x \) or the \( b_\eta \) is contained in the subgroup of \( H \) generated by the other generators and \( G \). For suppose that some \( b_\eta \in G \{G, x, b_\sigma: \sigma < \eta \} \). Then an equation \( b_\eta = w \) holds in \( H \) where \( w \) does not contain \( b_\eta \). We may suppose that \( w \) is \( R \)-reduced. Now any element of \( R \) contains only one \( b_\sigma \) generator or many occurrences of both \( b_\sigma \) and \( b_\eta \). Thus the equation \( w b_\eta^{-1} = 1 \) in \( H \) cannot hold.

The Torsion Theorem for small cancellation quotients of free products says that the only elements of finite order in \( H \) are conjugates of elements of \( G \), conjugates of powers of \( x \), and conjugates of powers of the \( b_\sigma \). Knowing all the elements of finite order is a key point in analyzing automorphisms of \( H \). Let \( \varphi \) be any automorphism of \( H \). We need only prove that, up to an inner automorphism, \( \varphi \) fixes \( x \) and all the \( b_\sigma \). Since \( \varphi(x) \) has order eleven, \( \varphi(x) \) is either a conjugate of an element of \( G \) or a conjugate of a power of \( x \). Following \( \varphi \) by an inner automorphism we may suppose that \( \varphi(x) \) is in a factor. First suppose that \( \varphi(x) = g \in G \). Now \( H = Gp(G, x, b_\sigma: \sigma < \lambda) \) but, by the remark above, \( x \not\in Gp(G, b_\eta: \sigma < \eta) \). Thus for some \( \eta \) we must have \( \varphi(b_\eta) = u g_1 u^{-1} \) with \( g_1 \in G \) or \( \varphi(b_\eta) = u b_\eta^k u^{-1} \) where \( u \) is \( R \)-reduced and contains \( x \). Following \( \varphi \) by another inner automorphism if necessary, we may assume that \( u \) does not begin with an element of \( G \). In the case that \( \varphi(b_\eta) = u b_\eta^k u^{-1} \) we have \( \varphi(s_\eta) = g u b_\eta^k u^{-1} \cdots (g u b_\eta^k u^{-1})^{160} \) which must be equal to the identity in \( H \). But the above expression is in free product normal form as written and cannot contain seven-tenths of an element of \( R \). Similarly, the case \( \varphi(b_\eta) = u g^{-1} u^{-1} \) leads to a contradiction. We conclude that \( \varphi(x) = x^j \) for some \( j \). With this fact a similar argument now establishes that each \( \varphi(b_\sigma) = b_\sigma^k \) for some choice of \( \tau \) and \( \eta \). But now application of \( \varphi \) to the relators \( s_\sigma \) shows that \( j \) and each \( k \) must be equal to one. For,
\[ \varepsilon(s_\sigma) = x^j b_\gamma^k (x^j b_\gamma^k)^{80} \cdots (x^j b_\gamma^k)^{160} \]
and the only way that the latter expression can contain a large part of a relator is to have \( j = k = 1 \). So we conclude that \( \varphi \) fixes \( x \) and permutes the \( b_\sigma \).

We now show that \( \varphi \) fixes the \( b_\sigma \). First of all, we show that if \( \sigma < \tau \), \( \varphi(b_\tau) = b_\gamma \), and \( \varphi(b_\tau) = b_\eta \) then \( \gamma < \eta \), that is, \( \varphi \) is order-preserving on subscripts. For, applying \( \varphi \) to \( t_{\sigma, \tau} \) we have
\[ \varphi(t_{\sigma, \tau}) = x b_\gamma x b_\eta x b_\eta^2 \cdots (x b_\gamma)^{80} \]
which must equal the indentity in $H$. The only element of $R$ which $\varphi(t_{\sigma,\tau})$ could contain a large part of is $t_{\gamma,\eta}$ which yields $\gamma < \eta$. But now we can conclude the proof, for, by the remark on the minimality of the generating set $\{x, b_{\sigma} : \sigma < \lambda\}$, all $b_{\sigma}$ must occur in the range of $\varphi$. If $\varphi$ does not fix every $b_{\sigma}$, let $\delta$ be the least subscript such that $\varphi(b_{\delta}) \neq b_{\delta}$. Then $\varphi(b_{\delta}) = b_{\eta}$ with $\eta > \delta$. Since $\varphi$ is order-preserving on subscripts, we have $b_{\delta} \notin \text{Gp}\{x, \varphi(b_{\sigma}) : \sigma < \lambda\}$ contradicting the fact that the latter set generates $H$. Thus $H$ is complete. 

Bibliography


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